On the "Universal" N=2 Supersymmetry of Classical Mechanics.

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Abstract

In this paper we continue the study of the geometrical features of a functional approach to classical mechanics proposed some time ago. In particular we try to shed some light on a N=2 "universal" supersymmetry which seems to have an interesting interplay with the concept of ergodicity of the system. To study the geometry better we make this susy local and clarify pedagogically several issues present in the literature. Secondly, in order to prepare the ground for a better understanding of its relation to ergodicity, we study the system on constant energy surfaces. We find that the procedure of constraining the system on these surfaces injects in it some local Grassmannian invariances and reduces the N=2 global susy to an N=1.

1 Introduction

In this paper we continue the study of a path-integral approach to Classical Mechanics (CM) started in ref. [1]. This path-integral is nothing else than the functional counterpart of an *operatorial* approach to CM pioneered by Koopman and von Neumann [2] in the 30's. This operatorial formulation is basically the one where the evolution is given by the Liouville operator.

In our approach we had a lot of extra variables besides those labelling the phase-space of the original mechanical system. It was discovered [1] [3] that these extra variables had a beautiful geometrical meaning. They were basically the basis for the forms and the tensor fields which one could build out of the tangent and cotangent bundles to phase-space. As a consequence our approach not only reproduced the Liouville operator of Koopman-von Neumann, which in geometrical terms [4] is the Hamiltonian vector field associated to the motion, but it generated automatically the entire Lie-derivative which is necessary for the motion of higher forms and tensors.

These extra variables were anyhow redundant and this redundancy was signalled by the presence of some "universal" symmetries which made a superalgebra known as Isp(2). The meaning of the charges associated to these symmetries was also intrinsically geometrical. In fact they turned out to be related to the exterior derivatives on phase-space and the form-number operator. The entire Cartan calculus on phase-space could be reproduced via our "universal" charges. All this will be briefly summarized in section 2 of this paper.

Beside these symmetries, it was later discovered [5] a new one which was actually a non-relativistic supersymmetry (susy). Those authors, anyhow, did not manage to understand the geometrical meaning of that susy and this paper is an attempt in that direction. That susy had [5] also a nice interplay with the concept of ergodicity [6] of the dynamical system under study and we think that it is crucial to get a better understanding of this interplay.

To tackle the first issue, i.e. the geometrical aspects of our susy, the direction we take is to make the susy local and study in detail what we obtain. We say "study in detail" because in the literature there were some strange statements [7] claiming to show that, at least for the supersymmetric QM of Witten [8], the lagrangian with local-susy was equivalent to the one with global susy. We shall show that it is not so. One should actually perform very carefully the full Dirac [9] procedure or, via path-integrals, the Faddeev procedure [10] or apply the BFV methodology [11] of handling systems with constraints. If one does that carefully it is easy to realize that the system with local susy has a different number of degrees of freedom than the one with global susy. The states themselves are restricted to the so called physical states by the presence of the local symmetry. It was this last step that was missing in ref. [7] and which led to the

wrong conclusion. We show all this in great details in section 3 of this paper.

The physical states condition and the BFV procedure are what lead us, in section 4 of the paper, to understand the geometrical meaning of the susy charges. They turn out to be an essential ingredient to restrict the forms to the so called equivariant ones [13] [14] [15]. The business of equivariant cohomology has popped out recently in the literature in connection with topological field theories [16]. Some attempts [17] had been done in the past of cooking up a BRS-BFV charge which would produce as physical states the equivariant ones but without showing from which local symmetry this BRS-BFV charge was coming from. Here we have filled that gap that means we have shown in details which local symmetry gives rise to a BFV charge whose physical states are the equivariant ones. We think that providing, as we have done here, all the details of the path-integral and local symmetry construction will be helpful to the readers even for just purely pedagogical reasons.

In section 5 of the paper we turn to the other aspect of our supersymmetry, that is its interplay with the concept of ergodicity [5]. To get a better grasp of this problem we realized long ago [5] that we had to formulate our functional approach on constant energy surfaces. So here somehow we reverse the procedure followed in the previous sections. There we basically had found that the local supersymmetries constrain the states and the motion to some hypersurfaces of our enlarged space, here instead we shall constrain by hand the system to move on some fixed hypersurfaces, the constant energy ones, and check what happens. We realize that in this formulation the energy plays the role of a coupling and it turns out to be associated to a tadpole term of the new lagrangian. Moreover we find that by constraining the system on constant energy surfaces we gain a *local* graded symmetry which is not anyhow a local susy. Regarding the *global* symmetries we lose part of the original global N=2 susy which is now reduced to an N=1. This fact, which may appear as a bad feature of the procedure, may actually turn out to be a virtue. In fact we shall have to study the interplay of ergodicity and susy by means of only one susy charge and not two as before. Anyhow the detailed study of this interplay will be left to future papers where we will concentrate more on dynamical issues and not on geometrical ones as we have done here. Some detailed calculations are confined to few appendices.

2 Review of the Functional Approach To Classical Mechanics.

We will briefly review here what is contained in ref. [1]. In those papers the authors gave a *path integral* formulation to CM. This may sound absurd but we should remember that whenever a theory has an *operatorial* formulation it has also a *path-integral* one.

Now CM has an operatorial formulation proposed long ago by Koopman and von-Neumann [2]. This operatorial approach is basically the one where the time evolution of phase-space distributions is governed by the *Liouville* operator or for higher forms by the *Lie derivative* of the Hamiltonian flow [4]. So, if there is an operatorial approach, there should also be a path-integral one which we will indicate from now on as CPI for **C**lassical **P**ath **I**ntegral.

In CM we have a $2n$-dimensional phase space $\mathcal{M}$ whose coordinates we call $\varphi^a(a = 1, \ldots 2n)$, i.e.: $\varphi^a = (q^1 \cdots q^n, p^1 \cdots p^n)$ and we indicate the Hamiltonian of the system as $H(\varphi)$ while the symplectic matrix is ω^{ab}. The equations of motion are then:

$$\dot{\varphi}^a = \omega^{ab} \frac{\partial H}{\partial \varphi^b} \tag{1}$$

We shall now introduce the *classical* analog, Z_{CM}, of the quantum generating functional:

$$Z_{CM}[J] = N \int \mathcal{D}\varphi \; \tilde{\delta}[\varphi(t) - \varphi_{cl}(t)] \; \exp \int J\varphi \, dt \tag{2}$$

where φ are the $\varphi^a \in \mathcal{M}$, φ_{cl} are the solutions of eq.(1), J is an external current and $\tilde{\delta}[\;\;]$ is a functional Dirac-delta which forces every path $\varphi(t)$ to sit on a classical one $\varphi_{cl}(t)$. There are all possible initial conditions integrated over in (2) and, because of this, one should be very careful in properly defining the measure of integration and the functional Dirac delta.

We should now check whether the path integral of eq. (2) leads to the well known operatorial formulation [2] of CM done via the Liouville operator and the Lie derivative. To do that let us first rewrite the functional Dirac delta in (2) as:

$$\tilde{\delta}[\varphi - \varphi_{cl}] = \tilde{\delta}[\dot{\varphi}^a - \omega^{ab}\partial_b H] \; \det[\delta^a_b \partial_t - \omega^{ac}\partial_c\partial_b H] \tag{3}$$

where we have used the analog of the relation $\delta[f(x)] = \frac{\delta[x - x_i]}{\left|\frac{\partial f}{\partial x}\right|_{x_i}}$.

The determinant which appears in (3) is always positive and so we can drop the modulus sign $|\;\;|$. The next step is to insert (3) in (2) and write the $\tilde{\delta}[\;\;]$ as a Fourier transform over some new variables λ_a, i.e.:

$$\tilde{\delta}\left[\dot{\varphi}^a - \omega^{ab}\frac{\partial H}{\partial \varphi^b}\right] = \int \mathcal{D}\lambda_a \; \exp \; i \int \lambda_a \left[\dot{\varphi}^a - \omega^{ab}\frac{\partial H}{\partial \varphi^b}\right] dt \tag{4}$$

Next we express the determinant $\det[\delta^a_b \partial_t - \omega^{ac}\partial_c\partial_b H]$ via Grassmannian variables

$\overline{c}_a, c^a$:

$$\det[\delta^a_b\partial_t - \omega^{ac}\partial_c\partial_b H] = \int \mathcal{D}c^a\mathcal{D}\overline{c}_a \;\exp\left[-\int \overline{c}_a[\delta^a_b\partial_t - \omega^{ac}\partial_c\partial_b H]c^b\; dt\right] \tag{5}$$

Inserting (4), (5) and (3) in (2) we get:

$$Z_{CM}[0] = \int \mathcal{D}\varphi^a\mathcal{D}\lambda_a\mathcal{D}c^a\mathcal{D}\overline{c}_a \;\exp\left[i\int dt\;\widetilde{\mathcal{L}}\right] \tag{6}$$

where $\widetilde{\mathcal{L}}$ is:

$$\widetilde{\mathcal{L}} = \lambda_a[\dot{\varphi}^a - \omega^{ab}\partial_b H] + i\overline{c}_a[\delta^a_b\partial_t - \omega^{ac}\partial_c\partial_b H]c^b \tag{7}$$

The variation of the action associated to this lagrangian gives the following equations of motion:

$$\dot{\varphi}^a - \omega^{ab}\partial_b H = 0 \tag{8}$$
$$[\delta^a_b\partial_t - \omega^{ac}\partial_c\partial_b H]c^b = 0 \tag{9}$$
$$\delta^a_b\partial_t\overline{c}_a + \overline{c}_a\omega^{ac}\partial_c\partial_b H = 0 \tag{10}$$
$$[\delta^a_b\partial_t + \omega^{ac}\partial_c\partial_b H]\lambda_a = -i\overline{c}_a\omega^{ac}\partial_c\partial_d\partial_b H c^d \tag{11}$$

From these equations we gather that $\widetilde{\mathcal{L}}$ leads to the same Hamiltonian equations for φ as H did and that c^b transforms under the Hamiltonian vector field [4] $h \equiv \omega^{ab}\partial_b H\partial_a$ as a *form* $d\varphi^b$ does.

The Hamiltonian $\widetilde{\mathcal{H}}$ associated to the lagrangian (7) is:

$$\widetilde{\mathcal{H}} = \lambda_a\omega^{ab}\partial_b H + i\overline{c}_a\omega^{ac}(\partial_c\partial_b H)c^b \tag{12}$$

and via some super-extended Poisson brackets (*EPB*) defined in the space $(\varphi^a, c^a, \lambda_a, \overline{c}_a)$ one can re-obtain the equations of motion (8)-(11). If we had considered all the variables $(\varphi^a, c^a, \lambda_a, \overline{c}_a)$ as configurational ones, then there would have been constraints among these variables and the associated momenta. In that case we would have had to adopt the Dirac procedure [9]. It will be explained in ref. [18] how this can be done and that the *EPB*-brackets above will be exactly the brackets produced by the Dirac method. These extended Poisson brackets are:

$$\{\varphi^a, \lambda_b\}_{EPB} = \delta^a_b \;\;;\;\; \{\overline{c}_b, c^a\}_{EPB} = -i\delta^a_b \tag{13}$$

All the other *EPB* are zero, in particular $\{\varphi^a, \varphi^b\}_{EPB} = 0$. Note from this relation that the *EPB* are not the standard Poisson brackets on $\mathcal{M}$ which would have given: $\{\varphi^a, \varphi^b\}_{PB} = \omega^{ab}$.

Since (6) is a path integral, one could also introduce the concept of *commutator* as Feynman did in the quantum case. If we define the graded commutator of two functions $O_1(t)$ and $O_2(t)$ as the expectation value $\langle \quad \rangle$ under our path integral of some time-splitting combinations of the functions themselves, as:

$$\langle [O_1(t), O_2(t)] \rangle \equiv \lim_{\epsilon \rightarrow 0} \langle O_1(t+\epsilon) O_2(t) \pm O_2(t+\epsilon) O_1(t) \rangle \tag{14}$$

then we get from (6) that the only commutators different from zero are:

$$\langle [\varphi^a, \lambda_b] \rangle = i\delta^a_b \quad ; \quad \langle [\overline{c}_b, c^a] \rangle = \delta^a_b. \tag{15}$$

Note that there is an isomorphism between the extended Poisson structure (13) and the graded commutator structure (15):

$$\{\cdot, \cdot\}_{EPB} \longrightarrow -i[\cdot, \cdot] \tag{16}$$

and we will always use the second one from now on. The commutator structure (15) allow us to "realize" λ_a and $\overline{c}_a$ as:

$$\lambda_a = -i\frac{\partial}{\partial \varphi^a} \quad ; \quad \overline{c}_a = \frac{\partial}{\partial c^a} \tag{17}$$

Now we have all the tools to turn the weight in (6) into an operator. For the moment let us take only the bosonic part of $\widetilde{\mathcal{H}}$:

$$\widetilde{\mathcal{H}}_B = \lambda_a \omega^{ab} \partial_b H \tag{18}$$

This one, via (17), turns into the operator:

$$\widehat{\widetilde{\mathcal{H}}}_B \equiv -i\omega^{ab} \partial_b H \partial_a \tag{19}$$

which is the Liouville operator of CM. If we had added the Grassmannian part to $\widetilde{\mathcal{H}}_B$ and inserted the operatorial representation (17) of $\overline{c}$, we would have got the Lie-derivative of the Hamiltonian flow as we shall show later . So this proves that the operatorial version of CM comes from a path-integral weight which is just a Dirac delta on the classical paths. Somehow this is the *classical* analogue of what Feynman did for *Quantum* Mechanics where he proved that the Schroedinger operator of evolution comes from a path-integral weight of the form $\exp{(iS)}$.

We have seen before that c^a transform as $d\varphi^a$, that is as the *basis* of generic forms $\alpha \equiv \alpha_a(\varphi) d\varphi^a$, but they also transform as the *components* of tangent vectors: $V^a(\varphi)\frac{\partial}{\partial \varphi^a}$.

The space whose coordinates are (φ^a, c^a) is called, in ref. [19], the *reverse-parity* tangent bundle and it is indicated as $\Pi T\mathcal{M}$. The "*reversed-parity*" specification is because the c^a are Grassmannian variables. As the $(\lambda_a, \bar{c}_a)$ are the "momenta" of the previous variables (see eq.(7)) we conclude that the $8n$ variables $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$ span the cotangent bundle to the reversed-parity tangent bundle: $T^\star(\Pi T\mathcal{M})$. So our super-space is a cotangent bundle and this is the reason why it has a Poisson structure which is the one we found via the CPI and indicated in eq. (13). For more details about this we refer the interested reader to ref. [3].

In the remaining part of this section we will show how to reproduce all the abstract Cartan calculus via our commutators and the Grassmannian variables. Let us first introduce five charges which are conserved under the $\widetilde{\mathcal{H}}$ of eq. (12) and which will play an important role in the Cartan calculus. They are:

$$Q_{BRS} \equiv ic^a\lambda_a \tag{20}$$
$$\overline{Q}_{BRS} \equiv i\bar{c}_a\omega^{ab}\lambda_b \tag{21}$$
$$Q_g \equiv c^a\bar{c}_a \tag{22}$$
$$K \equiv \frac{1}{2}\omega_{ab}c^ac^b \tag{23}$$
$$\overline{K} \equiv \frac{1}{2}\omega^{ab}\bar{c}_a\bar{c}_b \tag{24}$$

The ω_{ab} are the matrix elements of the inverse of ω^{ab}. These five charges make a superalgebra which we [1] called Isp(2) for inhomogeneous symplectic group. The reason for the name will be clear if we use a superspace as it was done in ref. [1].

Now since c^a transforms under the Hamiltonian flow as the basis $d\varphi^a$ of forms and $\bar{c}_a$ transforms as the basis of vector fields[1], (see eq. (10)), let us start building the following map, called "hat" map $\wedge$:

$$\alpha = \alpha_a d\varphi^a \;\overset{\wedge}{\longrightarrow}\; \widehat{\alpha} \equiv \alpha_a c^a \tag{25}$$
$$V = V^a\partial_a \;\overset{\wedge}{\longrightarrow}\; \widehat{V} \equiv V^a\bar{c}_a \tag{26}$$

It is actually a much more general map between forms α, antisymmetric tensors V and functions of $\varphi, c, \bar{c}$:

$$F^{(p)} = \frac{1}{p!}F_{a_1\cdots a_p}d\varphi^{a_1}\wedge\cdots\wedge d\varphi^{a_p} \;\overset{\wedge}{\longrightarrow}\; \widehat{F}^{(p)} \equiv \frac{1}{p!}F_{a_1\cdots a_p}c^{a_1}\cdots c^{a_p} \tag{27}$$
$$V^{(p)} = \frac{1}{p!}V^{a_1\cdots a_p}\partial_{a_1}\wedge\cdots\wedge\partial_{a_p} \;\overset{\wedge}{\longrightarrow}\; \widehat{V} \equiv \frac{1}{p!}V^{a_1\cdots a_p}\bar{c}_{a_1}\cdots\bar{c}_{a_p} \tag{28}$$

[1]Note that λ_a does not seem to transform as a vector field, eq. (11), even if it can be interpreted as $\frac{\partial}{\partial\varphi^a}$. The explanation of this fact is given in the second paper of ref. [3].

Once the correspondence (25)-(28) is extablished, we can easily find out what corresponds to the various Cartan operations like the exterior derivative d of a form, the interior contraction ι_V between a vector field V and a form F and the multiplication of a form by its form number [1] :

$$\begin{aligned} dF^{(p)} &\stackrel{\hat{}}{\longrightarrow} [Q_{BRS}, \widehat{F}^{(p)}] \qquad (29)\\ \iota_V F^{(p)} &\stackrel{\hat{}}{\longrightarrow} [\widehat{V}, \widehat{F}^{(p)}] \qquad (30)\\ pF^{(p)} &\stackrel{\hat{}}{\longrightarrow} [Q_g, \widehat{F}^{(p)}] \qquad (31)\end{aligned}$$

where Q_{BRS}, Q_g are the charges of (20)-(22). In the same manner we can translate in our language the usual mapping [4] between vector fields V and forms $V^\flat$ realized by the symplectic 2-form $\omega(V,0) \equiv V^\flat$, or the inverse operation of building a vector field $\alpha^\sharp$ out of a form: $\alpha = (\alpha^\sharp)^\flat$. These operations can be translated in our formalism as follows:

$$\begin{aligned} V^\flat &\stackrel{\hat{}}{\longrightarrow} [K, \widehat{V}] \qquad (32)\\ \alpha^\sharp &\stackrel{\hat{}}{\longrightarrow} [\overline{K}, \hat{\alpha}] \qquad (33)\end{aligned}$$

where again $K, \overline{K}$ are the charges (23)-(24). We can also translate the standard operation of building an Hamiltonian vector field, indicated as $(df)^\sharp$, out of a function $f(\varphi)$, and also the Poisson brackets between two functions f and g:

$$\begin{aligned} (df)^\sharp &\stackrel{\hat{}}{\longrightarrow} [\overline{Q}_{BRS}, f] \qquad (34)\\ \{f,g\}_{PB} = df[(dg)^\sharp] &\stackrel{\hat{}}{\longrightarrow} [[[f, Q_{BRS}], \overline{K}], [[[g, Q_{BRS}], \overline{K}], K]] \qquad (35)\end{aligned}$$

The next thing to do is to translate the concept of Lie derivative which is defined as:

$$\mathcal{L}_V = d\iota_V + \iota_V d \qquad (36)$$

It is easy to prove that:

$$\mathcal{L}_V F^{(p)} \stackrel{\hat{}}{\longrightarrow} i[\widetilde{\mathcal{H}}_V, \widehat{F}^{(p)}] \qquad (37)$$

where $\widetilde{\mathcal{H}}_V = \lambda_a V^a + i\bar{c}_a \partial_b V^a c^b$. Note that, for $V^a = \omega^{ab}\partial_b H$, the $\widetilde{\mathcal{H}}_V$ becomes the $\widetilde{\mathcal{H}}$ of (12). This confirms that the full $\widetilde{\mathcal{H}}$ of eq. (12) is the Lie derivative of the Hamiltonian flow . One last point to notice is that any $\widetilde{\mathcal{H}}$ associated to an Hamiltonian flow can be written as:

$$\widetilde{\mathcal{H}} = -i[Q_{BRS}[\overline{Q}_{BRS}, H]] \qquad (38)$$

where H is the 0-form out of which we build the Hamiltonian vector field (via eq.(34)) which enters $\widetilde{\mathcal{H}}$. The structure of eq.(38), that is a double commutator with both BRS

and antiBRS charges, is really what embodies both the Lie-derivative structure and the Hamiltonian vector field structure. A Lie derivative of a *vector* field (and not of an Hamiltonian one) would have been expressed only as a single commutator (with respect to the BRS charge) and not as a double commutator. The first commutator with the $\overline{Q}_{BRS}$ in (38) embodies the Hamiltonian vector structure given by eq.(34).

There are many other structures in symplectic differential geometry which can be translated in our formalism and the interested reader can look them up in [3].
Besides the five conserved charges listed in eqs.(20)-(24) there are two more [5]:

$$N_H = c^a \partial_a H \qquad ; \qquad \overline{N}_H = \overline{c}_a \omega^{ab} \partial_b H \tag{39}$$

Combining them with the Q_{BRS} and $\overline{Q}_{BRS}$ of (20)-(21) we get the following two extra conserved charges:

$$Q_H \equiv Q_{BRS} - \beta N_H \qquad ; \qquad \overline{Q}_H \equiv \overline{Q}_{BRS} + \beta \overline{N}_H \tag{40}$$

where β is a dimensional parameter. These two new charges are true supersymmetry charges, in fact we have:

$$[Q_H, \overline{Q}_H] = 2i\beta\widetilde{\mathcal{H}}. \tag{41}$$

We have not studied geometrically these charges in as many details as we did for the Isp(2) charges of eqs.(20)-(24). That is what we plan to do in the next two sections.

3 Gauging the Global Susy Invariance.

The direction we take to study the geometrical structures behind the supersymmetric charges above is to build a lagrangian where these symmetries are local. The standard procedure we use is known in the literature [20] as the Noether method. It basically consists in finding the exact form of the extra terms generated by *local* variations of the original lagrangian which had only the global invariances. These extra terms, by Noether's theorem, are basically the derivatives of the infinitesimal parameters multiplied by the generators. The trick then is to add to the original lagrangian a piece made of an *auxiliary field* multiplied by the generator. We can then impose that this auxiliary field transform in such a manner as to cancel the extra variations of the lagrangian mentioned above.

As the susy charges are built out of the $Q_{BRS}, \overline{Q}_{BRS}, N_H, \overline{N}_H$ let us build the *local* variations generated by each of these charges on the variables $(\varphi^a, c^a, \lambda_a, \overline{c}_a)$. If we indicate with X one of those four operators and with $(\cdot)$ any of the variables $(\varphi^a, c^a, \lambda_a, \overline{c}_a)$, then by a local variation, δ_X^{loc}, we indicate the operation: $\delta_X^{loc} \equiv [\varepsilon(t)X, (\cdot)]$ where now

the Grassmannian parameter ε is dependent on t. These four variations are indicated below:

$$\delta_{Q}^{loc}\equiv\begin{cases}\delta\varphi^{a}=\epsilon(t)c^{a}\\ \delta c^{a}=0\\ \delta\overline{c}_{a}=i\epsilon(t)\lambda_{a}\\ \delta\lambda_{a}=0\end{cases}\qquad\delta_{\overline{Q}}^{loc}\equiv\begin{cases}\delta\varphi^{a}=-\overline{\epsilon}(t)\omega^{ab}\overline{c}_{b}\\ \delta c^{a}=i\overline{\epsilon}(t)\omega^{ab}\lambda_{b}\\ \delta\overline{c}_{a}=0\\ \delta\lambda_{a}=0\end{cases}\tag{42}$$

$$\delta_{N}^{loc}\equiv\begin{cases}\delta\varphi^{a}=0\\ \delta c^{a}=0\\ \delta\overline{c}_{a}=\epsilon(t)\partial_{a}H\\ \delta\lambda_{a}=i\epsilon(t)c^{b}\partial_{b}\partial_{a}H\end{cases}\qquad\delta_{\overline{N}}^{loc}\equiv\begin{cases}\delta\varphi^{a}=0\\ \delta c^{a}=\overline{\epsilon}(t)\omega^{ab}\partial_{b}H\\ \delta\overline{c}_{a}=0\\ \delta\lambda_{a}=i\overline{\epsilon}(t)\overline{c}_{d}\omega^{db}\partial_{b}\partial_{a}H.\end{cases}\tag{43}$$

We could have used four different parameters for the four different charges (as we will do later on) but here we limit ourselves just to two: $\epsilon(t)$ and $\overline{\epsilon}(t)$. The local susy variations associated to the two susy charges of eq. (40) are :

$$\begin{cases}\delta_{Q_H}^{loc}=\delta_{Q}^{loc}-\beta\delta_{N}^{loc}\\ \delta_{\overline{Q}_H}^{loc}=\delta_{\overline{Q}}^{loc}+\beta\delta_{\overline{N}}^{loc}.\end{cases}\tag{44}$$

It is now straightforward to check that the local susy variations of the lagrangian $\widetilde{\mathcal{L}}$ in (7) gives the following results:

$$\delta_{Q_H}^{loc}\widetilde{\mathcal{L}}=-i\dot{\epsilon}Q_{H}+(t.d.)\tag{45}$$

and

$$\delta_{\overline{Q}_H}^{loc}\widetilde{\mathcal{L}}=-i\dot{\overline{\epsilon}}\overline{Q}_{H}+(t.d.).\tag{46}$$

With $(t.d.)$ we indicate total derivative terms. They turn into surface terms in the action and they disappear if we require that $\epsilon(t)$ and $\overline{\epsilon}(t)$ be zero at the end points of integrations as we will do from now on. To do things in a cleaner manner we should have actually checked the invariance using the integrated charge as explained in appendix A. Anyhow from eq.(45) and (46) we see that the lagrangian does not change by a total derivative so the two local susy transformations are not symmetries and we have to modify the lagrangian to find another one which is invariant. If we find it, then it must also be invariant [20] under the composition of two local susy transformations which we can prove (see appendix B) to be the sum of a local supersymmetry transformation plus a *local* time-translation generated by $\widetilde{\mathcal{H}}$. This last one is not a symmetry of $\widetilde{\mathcal{L}}$ and the lagrangian changes by a term proportional to $\widetilde{\mathcal{H}}$ multiplied by the time-derivative

of the symmetry parameter, exactly as the Noether theorem requires. The trick [20] to get the invariance is to add to $\widetilde{\mathcal{L}}$ some auxiliary fields multiplied by the charges under which $\widetilde{\mathcal{L}}$ is not invariant. In our case the complete lagrangian turns out to be :

$$\widetilde{\mathcal{L}}_{susy} \equiv \widetilde{\mathcal{L}} + \overline{\psi} Q_H + \psi \overline{Q}_H + g\widetilde{\mathcal{H}}, \tag{47}$$

where $g(t), \psi(t), \overline{\psi}(t)$ are three new fields (the last two of Grassmannian nature) whose variations under the local susy will be determined by the requirement that $\widetilde{\mathcal{L}}_{susy}$ be invariant under the local susy variations of eq.(44). In detail we get:

$$\delta_{Q_H}\widetilde{\mathcal{L}}_{susy} = -i\dot{\epsilon} Q_H + (\delta_{Q_H} g)\widetilde{\mathcal{H}} + (\delta_{Q_H}\overline{\psi})Q_H + (\delta_{Q_H}\psi)\overline{Q}_H + \psi(2i\epsilon\beta\widetilde{\mathcal{H}}) \tag{48}$$

and we see that the following transformations of the variables $g, \psi, \overline{\psi}$ make $\widetilde{\mathcal{L}}_{susy}$ invariant under the local transformation associated to Q_H:

$$\begin{cases} \delta_{Q_H}\overline{\psi} = i\dot{\epsilon} \\ \delta_{Q_H}\psi = 0 \\ \delta_{Q_H} g = +2i\epsilon\beta\psi. \end{cases} \tag{49}$$

For the variation under $\overline{Q}_H$ we get:

$$\delta_{\overline{Q}_H}\widetilde{\mathcal{L}}_{susy} = -i\dot{\overline{\epsilon}}\overline{Q}_H + (\delta_{\overline{Q}_H} g)\widetilde{\mathcal{H}} + (\delta_{\overline{Q}_H}\overline{\psi})Q_H + (\delta_{\overline{Q}_H}\psi)\overline{Q}_H + \overline{\psi}(2i\overline{\epsilon}\beta\widetilde{\mathcal{H}}) \tag{50}$$

and we see that the following transformations of the variables $g, \psi, \overline{\psi}$ make $\widetilde{\mathcal{L}}_{susy}$ invariant under the local transformation associated to $\overline{Q}_H$:

$$\begin{cases} \delta_{\overline{Q}_H}\overline{\psi} = 0 \\ \delta_{\overline{Q}_H}\psi = i\dot{\overline{\epsilon}} \\ \delta_{\overline{Q}_H} g = +2i\overline{\epsilon}\beta\overline{\psi}. \end{cases} \tag{51}$$

Last we should check how $\widetilde{\mathcal{L}}$ changes under a local time-reparametrization. We have to do that because this reparametrization appears in the composition of two local susy transformations (Appendix B). The action of the local time reparametrization on the variables $(\varphi^a, c^a, \lambda_a, \overline{c}_a)$ is listed in formula (B.9) of appendix B. Under those trasformations we can easily prove that

$$\delta\widetilde{\mathcal{L}} = -i\dot{\eta}\widetilde{\mathcal{H}} \tag{52}$$

where $\eta(t)$ is the time-dependent parameter of the transformation. Let us now use this result in the variation of $\widetilde{\mathcal{L}}_{susy}$ under time reparametrization:

$$\delta\widetilde{\mathcal{L}}_{susy} = -i\dot{\eta}\widetilde{\mathcal{H}} + \delta\overline{\psi} Q_H + \delta\psi\overline{Q}_H + \delta g\widetilde{\mathcal{H}}. \tag{53}$$

We immediately notice that $\widetilde{\mathcal{L}}_{susy}$ is invariant under the local-time reparametrization if we transform the variables $(\psi, \overline{\psi}, g)$ as follows

$$\begin{cases} \delta\psi = \delta\overline{\psi} = 0 \\ \delta g = i\dot{\eta}. \end{cases} \tag{54}$$

So, putting together all the three local symmetries, (49), (51) and (54), we can say that $\widetilde{\mathcal{L}}_{susy}$ (if we choose $\beta = 1$ in eqs. (44)-(51)) has the following local invariance:

$$\begin{cases} \delta\psi = i\dot{\overline{\epsilon}} \\ \delta\overline{\psi} = i\dot{\epsilon} \\ \delta g = i\dot{\eta} + 2i(\overline{\epsilon}\overline{\psi} + \epsilon\psi). \end{cases} \tag{55}$$

It is not the first time that one-dimensional systems with local-susy have been built. The first work was the classic one of Brink et al. [20]. Later on people [7] have played with the supersymmetric Quantum Mechanical model (SUSY-QM) of Witten [8] turning its global susy into a local invariance. Regarding this model, the author of ref. [7] pretended to show that the locally supersymmetric quantum mechanics was equivalent to the standard SUSY QM with only global invariance. The proof was based on the fact that, via the analog of the transformations (55), it is possible to bring the variables $(\psi, \overline{\psi}, g)$ to zero and so, looking at eq.(47), this would imply that we can turn $\widetilde{\mathcal{L}}_{susy}$ into $\widetilde{\mathcal{L}}$. This kind of reasoning is *misleading*. In fact, while it is easy to check (see Appendix D) that it is possible to bring the $(\psi, \overline{\psi}, g)$ to zero via the transformations (55), it should be remembered that the starting point was a gauge theory, $\widetilde{\mathcal{L}}_{susy}$, and the value zero for the variables $(\psi, \overline{\psi}, g)$ is equivalent to a particular choice of gauge-fixing. Anyhow, the *physical* theory has to be gauge-fixing independent and this is achieved [11] by restricting the physical states via the BRS charge associated to the local symmetries. So in the gauge-fixing where the $(\psi, \overline{\psi}, g)$ are zero the locally-supersymmetric QM [7] has the same action as the globally supersymmetric theory [8] but we have to restrict the states to the physical ones which are basically those annihilated by the symmetry charges. So the two systems, the one with global susy and the one with local susy, are not equivalent even if they seem to be so in a particular gauge-fixing. Their Hilbert spaces are different even if the dynamics, in a particular gauge-fixing, is the same. Moreover, even at the level of counting of degrees of freedom we shall show that, while $\widetilde{\mathcal{L}}$ has $8n$ independent variables, $\widetilde{\mathcal{L}}_{susy}$ has $8n - 6$. To do this analysis we should go through the business of studying the constraints associated to the local symmetries of $\widetilde{\mathcal{L}}_{susy}$ as we are going to do in what follows.

The standard procedure is the one of Dirac [9] which we will follow here in detail.

Looking at $\widetilde{\mathcal{L}}_{susy}$ we see that the primary constraints are:

$$\begin{cases} \Pi_{\psi} = 0 \\ \Pi_{\overline{\psi}} = 0 \\ \Pi_{g} = 0 \end{cases} \tag{56}$$

where Π_{ψ}, $\Pi_{\overline{\psi}}$ and Π_{g} are the momenta associated to ψ, $\overline{\psi}$ and g. The *canonical* Hamiltonian [9] is then the following:

$$\widetilde{\mathcal{H}}_{can.} = \widetilde{\mathcal{H}}_{susy} = \widetilde{\mathcal{H}} - \psi \overline{Q}_{H} - \overline{\psi} Q_{H} - g\widetilde{\mathcal{H}} \tag{57}$$

while the *primary* [9] (or total) Hamiltonian is:

$$\widetilde{\mathcal{H}}_{P} = (1-g)\widetilde{\mathcal{H}} - \psi \overline{Q}_{H} - \overline{\psi} Q_{H} + u_1 \Pi_{\psi} + u_2 \Pi_{\overline{\psi}} + u_3 \Pi_{g} \tag{58}$$

and it is obtained by adding to $\widetilde{\mathcal{H}}_{can.}$ the primary constraints (56) via the Lagrange multipliers u_1, u_2, u_3. Next we have to impose that the primary constraints do not change under the time evolution, i.e:

$$[\Pi_{\psi}, \widetilde{\mathcal{H}}_{P}] = 0, \quad [\Pi_{\overline{\psi}}, \widetilde{\mathcal{H}}_{P}] = 0, \quad [\Pi_{g}, \widetilde{\mathcal{H}}_{P}] = 0. \tag{59}$$

We have used commutators here but it would have been more correct to use the Extended-Poisson-Brackets. We did that just because the two structures are isomorphic as explained in eq.(16). In particular the (graded) commutators we need in (59) are

$$[\Pi_{\psi}, \psi] = 1, \quad [\Pi_{\overline{\psi}}, \overline{\psi}] = 1, \quad [\Pi_{g}, g] = -i. \tag{60}$$

Using them we get from (59) the following set of *secondary* [9] constraints:

$$\begin{cases} \overline{Q}_{H} = 0 \\ Q_{H} = 0 \\ \widetilde{\mathcal{H}} = 0. \end{cases} \tag{61}$$

At this point the careful reader could ask which are the operators generating the full set of transformations (55), expecially the last one. It does not seem that they are generated by the operators of eqs. (61) and (56). Actually the answer to this question is rather subtle and tricky [11] and is given in full details in appendix C.

Having clarified this point, we can go on with our procedure. We have now to require that also the secondary constraints (61) do not change under time evolution using as operator of evolution always the primary Hamiltonian $\widetilde{\mathcal{H}}_{P}$ as explained in ref. [9]. It is easy to realize that in our case we do not generate further constraints with this

procedure and that, at the same time, we do not determine the Lagrange multipliers. The fact that the Lagrange multipliers are all left undetermined is a signal that the constraints are first class [9] as it is easy to check by doing the commutators among all the six constraints (61) and (56). Being them first class, one has to introduce six gauge-fixings which will be used to determine the Lagrange multipliers [9].

The gauge-fixings, let us call them χ_i, must have a non-zero commutator with the associated gauge generator. For the three constraints of eq.(56) three suitable gauge-fixings can be:

$$\psi - \psi_0 = 0, \quad \overline{\psi} - \overline{\psi}_0 = 0, \quad g - g_0 = 0, \tag{62}$$

where ψ_0, $\overline{\psi}_0$ and g_0 are three fixed functions. It is easy to check that each of them does not commute with its associated generator. The gauge-fixing $\psi_0 = \overline{\psi}_0 = g_0 = 0$ is among the admissible ones, in the sense that there is a gauge transformation which brings any configuration into this one as shown in appendix D. In this gauge fixing we get that the $(\psi, \overline{\psi}, g)$ variables disappear from the $\widetilde{\mathcal{L}}_{susy}$ and so $\widetilde{\mathcal{L}}_{susy}$ apparently is reduced to $\widetilde{\mathcal{L}}$. Of course, as we said earlier, this should not mislead us to think that the physics of $\widetilde{\mathcal{L}}_{susy}$ is the same as the one of $\widetilde{\mathcal{L}}$. In fact at the Hamiltonian level, even if $\widetilde{\mathcal{H}}_{susy}$ is reduced to $\widetilde{\mathcal{H}}$ by the gauge-fixing, the Poisson brackets for the two systems are different. For the one with local symmetries the Poisson brackets are the Dirac ones which, given two observables O_1 and O_2, are built as:

$$\{O_1, O_2\}_{DB} = \{O_1, O_2\} - \{O_1, G_i\}(C^{-1})^{ij}\{G_j, O_2\}, \tag{63}$$

We have indicated with G_i any of the six first class constraints of eq.(61)(56), and the matrix C_{ij} has its elements built as $\{G_i, \chi_j\}$ where χ_i are the six gauge-fixings associated to the constraints G_i. The brackets entering the expressions on the RHS of (63) are the standard Extended Poisson Brackets of eq.(13). If the dynamics is the one of a system with global susy only, that is one whose Hamiltonian is really $\widetilde{\mathcal{H}}$ from the beginning, then the Poisson brackets among the same two observables O_1 and O_2 would be $\{O_1, O_2\}$ and it is clear that

$$\{O_1, O_2\} \neq \{O_1, O_2\}_{DB}. \tag{64}$$

This explains why, even if the Hamiltonians of the two systems (the one with local symmetries and the one with global ones) are the same (in some gauge-fixings), the two dynamics are anyhow different because they are ruled by different Poisson brackets.

Also the counting of the degrees of freedom indicates that the systems have different numbers of degrees of freedom. The one with global symmetries, and lagrangian $\widetilde{\mathcal{L}}$, has just the variables $(\varphi^a, c^a, \lambda_a, \overline{c}_a)$ which are $8n$. The one with local symmetries, $\widetilde{\mathcal{L}}_{susy}$,

has the variables $(\varphi^a, c^a, \lambda_a, \overline{c}_a)$ which are $8n$, plus the three gauge variables $(\psi, \overline{\psi}, g)$ and the relative momenta for a total of 6, minus the 6 constraints of eq.(61)(56), minus the 6 gauge fixings χ_i, for a total of $8n-6$ variables. This is the correct counting of variables as explained in ref. [11]. So even from this we realize that the two systems are different.

As we said at the beginning, our path-integral is actually the counterpart of the operatorial version of CM proposed by Koopman and von Neumann [2] and if we adopt this operatorial version we should use the commutators derived in (15). This operatorial version of course could be adopted also for the dynamics with local symmetries associated to $\widetilde{\mathcal{H}}_{susy}$. In the operatorial formulation we have a Hilbert space but we know that, for a system with local symmetries, the Hilbert space is restricted to the *physical states*. The selection of these states is done by a BRS-BFV charge [11] associated to the gauge-symmetries of the system. This BRS-BFV charge of course has nothing to do with the Q_{BRS} of eq.(20). The construction of the BRS-BFV charge for our $\widetilde{\mathcal{L}}_{susy}$ goes as follows [11]. First we should introduce a pair of gauge ghost-antighosts for each gauge generator. As our gauge generators are

$$G_i = (\Pi_{\overline{\psi}}, \Pi_{\psi}, \Pi_g, Q_H, \overline{Q}_H, \widetilde{\mathcal{H}}) \tag{65}$$

the ghost-antighosts are twelve and can be indicated as:

$$\begin{aligned} \eta^i &= (\eta_{\overline{\psi}}, \eta_{\psi}, \eta_g, \eta_H, \overline{\eta}_H, \widetilde{\eta}_H) \\ \mathcal{P}_i &= (\mathcal{P}_{\overline{\psi}}, \mathcal{P}_{\psi}, \mathcal{P}_g, \mathcal{P}_H, \overline{\mathcal{P}}_H, \widetilde{\mathcal{P}}_H). \end{aligned} \tag{66}$$

The general BRS-BFV charge [11] is then[2]:

$$\Omega_{BRS} = \eta^i G_i - \frac{1}{2}(-)^{\epsilon_i}\eta^i\eta^j C^k_{ji}\mathcal{P}_k, \tag{67}$$

where ε_i is the Grassmannian grading of the constraints G_i and C^k_{ij} are the structure constants of the algebra of our constraints. In our case this algebra is:

$$\begin{aligned} &[Q_H, \overline{Q}_H] = 2i\widetilde{\mathcal{H}} \\ &[Q_H, Q_H] = [\overline{Q}_H, \overline{Q}_H] = [Q_H, \widetilde{\mathcal{H}}] = [\overline{Q}_H, \widetilde{\mathcal{H}}] = 0 \end{aligned} \tag{68}$$

where we have put $\beta = 1$ with respect to eq.(41).
It is now easy to work out the BRS-BFV charge for our local susy invariance:

$$\Omega^{(susy)}_{BRS} = \eta_{\overline{\psi}}\Pi_{\overline{\psi}} + \eta_{\psi}\Pi_{\psi} + \eta_g\Pi_g + \eta_H Q_H + \overline{\eta}_H\overline{Q}_H + \widetilde{\eta}_H\widetilde{\mathcal{H}} - 2i\eta_H\overline{\eta}_H\widetilde{\mathcal{P}}_H \tag{69}$$

[2]The graded commutators among the ghosts of (66) are $[\eta^i, \mathcal{P}_j] = \delta^i_j$

Note that it contains terms with three ghosts and so it is hard to see how it works on the states. These terms with three ghosts are there because the generators are not in involution. As it is explained in ref. [11] in case the constraints G_i are not in involution, one can build some new ones $F_i = a_i^j G_j$ which are in involution. In our case the F_i generators, replacing the Q_H and $\overline{Q}_H$, can be easily worked out and they are:

$$\begin{cases} Q_A \equiv (Q_H + \psi\widetilde{\mathcal{H}}) \\ Q_B \equiv (\overline{Q}_H - 2i\Pi_\psi) \end{cases} \tag{70}$$

while the other F_i are the same as the G_j. The associated BRS-BFV charge is then

$$\Omega_{BRS}^{(F)} = \eta_{\overline{\psi}}\Pi_{\overline{\psi}} + \eta_\psi\Pi_\psi + \eta_g\Pi_g + \eta_A Q_A + \eta_B Q_B + \widetilde{\eta}_H\widetilde{\mathcal{H}}. \tag{71}$$

We have called η_A, η_B the BFV ghosts associated to Q_A, Q_B. Note that this $\Omega_{BRS}^{(F)}$ does not contain terms with three ghosts. The *physical states* are then defined as

$$\Omega_{BRS}^{(F)} \mid \text{phys}\rangle = 0 \tag{72}$$

and, by following ref. [11], we can easily show that (72) is equivalent to the following six constraints:

$$(1)\begin{cases} \Pi_{\overline{\psi}} \mid \text{phys}\rangle = 0 \\ \Pi_\psi \mid \text{phys}\rangle = 0 \\ \Pi_g \mid \text{phys}\rangle = 0 \end{cases} \qquad (2)\begin{cases} Q_A \mid \text{phys}\rangle = 0 \\ Q_B \mid \text{phys}\rangle = 0 \\ \widetilde{\mathcal{H}} \mid \text{phys}\rangle = 0. \end{cases} \tag{73}$$

The set (1) above means that the physical states must be independent of $(\psi, \overline{\psi}, g)$ that means independent of any choice of gauge fixing. The set (2) instead (combined with some of the conditions from the set (1)) is equivalent to the following conditions:

$$\begin{aligned} &Q_H \mid \text{phys}\rangle = 0 \\ &\overline{Q}_H \mid \text{phys}\rangle = 0 \\ &\widetilde{\mathcal{H}} \mid \text{phys}\rangle = 0. \end{aligned} \tag{74}$$

We can summarize it by saying that, even if in some gauge-fixing the $\widetilde{\mathcal{H}}_{susy}$ is the same as $\widetilde{\mathcal{H}}$, the dynamics of the first is restricted to a subset (given by eq.(74)) of the full Hilbert space while the dynamics of $\widetilde{\mathcal{H}}$ is not restricted. This is what is not spelled out correctly in ref. [7]. The author may have been brought to the wrong conclusions not only because he did not consider the correct Hilbert space but also by the following fact that we like to draw to the attention of the reader. The three quantities $Q_H, \overline{Q}_H, \widetilde{\mathcal{H}}$ entering the constraints (61) are actually constants of motion in the space $(\varphi^a, c^a, \lambda_a, \overline{c}_a)$. So fixing them, as the constraints do, is basically fixing a set of initial conditions. The hypersurface defined by (61) is the subspace of $(\varphi^a, c^a, \lambda_a, \overline{c}_a)$ where

the motion takes place and it takes place with the same dynamics as in $(\varphi^a, c^a, \lambda_a, \overline{c}_a)$. If the constraint surfaces were not made by constants of motion, then the dynamics would have to be modified to force the particle to move on them, but this is not the case here.

Before concluding let us notice that the analysis we have made seems to tell us that, by forcing the particle to move on some hypersurfaces fixed by particular values of the constants of motion, we generate a dynamics with local-invariances. We will explore this issue further in later sections of the paper.

4 New Local Susy Invariance and Equivariance

We have seen in section 2 that, besides the susy, there are other global invariances of $\widetilde{\mathcal{H}}$. We are tempted to gauge all of them and see what comes out. In this paper we will limit ourselves to study what happens when we gauge separately the Q_{BRS}, $\overline{Q}_{BRS}$, N_H, $\overline{N}_H$ of eqs. (20), (21) and (39). The local variation of $\widetilde{\mathcal{L}}$ under these four combined gauge transformations is

$$
\begin{aligned}
\delta^{loc.}\widetilde{\mathcal{L}} &= [\epsilon Q_{BRS} + \overline{\epsilon}\overline{Q}_{BRS} + \eta N_H + \overline{\eta}\overline{N}_H, \widetilde{\mathcal{L}}] \\
&= -i\dot{\epsilon}Q_{BRS} - i\dot{\overline{\epsilon}}\overline{Q}_{BRS} - i\dot{\eta}N_H - i\dot{\overline{\eta}}\overline{N}_H
\end{aligned} \tag{75}
$$

where $(\epsilon(t), \overline{\epsilon}(t), \eta(t), \overline{\eta}(t))$ are four different Grassmannian gauge parameters. From equation (75) one could be tempted to propose the following as *extended* lagrangian invariant under the local symmetries above:

$$
\widetilde{\mathcal{L}}_{ext.} \equiv \widetilde{\mathcal{L}} + \alpha(t)Q_{BRS} + \overline{\alpha}(t)\overline{Q}_{BRS} + \beta(t)N_H + \overline{\beta}(t)\overline{N}_H \tag{76}
$$

where $(\alpha(t), \overline{\alpha}(t), \beta(t), \overline{\beta}(t))$ are four Grassmannian gauge-fields which we could transform in a proper way in order to make $\widetilde{\mathcal{L}}_{ext.}$ invariant. This is actually impossible whatever transformation we envision for the gauge-fields. In fact, as we did in the case of the susy of the previous section, we should consider what happens when we compose two of the local symmetries of eq.(75). This information is given by the following commutators [1]:

$$
[Q_{BRS}, \overline{N}_H] = i\widetilde{\mathcal{H}} \qquad ; \qquad [\overline{Q}_{BRS}, N_H] = -i\widetilde{\mathcal{H}} \tag{77}
$$

This basically tells us that we should add to the lagrangian $\widetilde{\mathcal{L}}_{ext.}$ of eq.(76) an extra gauge field $g(t)$ and an extra gauge generator[3] $\widetilde{\mathcal{H}}$:

[3]We will indicate with greek letters the gauge fields associated to Grassmannian generators and with latin letters the one associated to bosonic generators.

$$\widetilde{\mathcal{L}}_{ext.} = \widetilde{\mathcal{L}} + \alpha(t)Q_{\scriptscriptstyle BRS} + \overline{\alpha}(t)\overline{Q}_{\scriptscriptstyle BRS} + \beta(t)N_{\scriptscriptstyle H} + \overline{\beta}(t)\overline{N}_{\scriptscriptstyle H} + g(t)\widetilde{\mathcal{H}} \tag{78}$$

Doing now an extended gauge transformation like in (75) we get:

$$\begin{aligned} \delta_{loc.}\widetilde{\mathcal{L}}_{ext.} = & - i\dot{\epsilon}Q_{\scriptscriptstyle BRS} - i\dot{\overline{\epsilon}}\overline{Q}_{\scriptscriptstyle BRS} - i\dot{\eta}N_{\scriptscriptstyle H} - i\dot{\overline{\eta}}\overline{N}_{\scriptscriptstyle H} + (\delta\alpha)Q_{\scriptscriptstyle BRS} \\ & + i\alpha\overline{\eta}\widetilde{\mathcal{H}} + (\delta\overline{\alpha})\overline{Q}_{\scriptscriptstyle BRS} - i\overline{\alpha}\eta\widetilde{\mathcal{H}} + (\delta\beta)N_{\scriptscriptstyle H} \\ & - i\beta\overline{\epsilon}\widetilde{\mathcal{H}} + (\delta\overline{\beta})\ \overline{N}_{\scriptscriptstyle H} + i\overline{\beta}\epsilon\widetilde{\mathcal{H}} + (\delta g)\widetilde{\mathcal{H}} \end{aligned} \tag{79}$$

and from this it is easy to see that $\widetilde{\mathcal{L}}_{ext.}$ is invariant if the gauge-fields $(\alpha, \overline{\alpha}, \beta, \overline{\beta}, g)$ are transformed as follows:

$$\begin{cases} \delta\alpha = i\dot{\epsilon} \\ \delta\overline{\alpha} = i\dot{\overline{\epsilon}} \\ \delta\beta = i\dot{\eta} \\ \delta\overline{\beta} = i\dot{\overline{\eta}} \\ \delta g = i\overline{\alpha}\eta - i\alpha\overline{\eta} + i\beta\overline{\epsilon} - i\overline{\beta}\epsilon \end{cases} \tag{80}$$

From these transformations we notice that, for some choice of the gauge-fields and of the gauge-transformations, we do not need to have the $g\widetilde{\mathcal{H}}$ in the $\widetilde{\mathcal{L}}_{ext.}$ The first choice is $\alpha = \overline{\alpha} = \epsilon = \overline{\epsilon} = 0$ which, from eq.(80), implies that we can choose $g(t) = 0$. The $\widetilde{\mathcal{L}}_{ext.}$ is then

$$\widetilde{\mathcal{L}}_{\scriptscriptstyle N} \equiv \widetilde{\mathcal{L}} + \beta(t)N_{\scriptscriptstyle H} + \overline{\beta}(t)\overline{N}_{\scriptscriptstyle H} \tag{81}$$

The second choice, which also implies that we can choose $g(t) = 0$, is $\beta = \overline{\beta} = \eta = \overline{\eta} = 0$ and this would lead to the following lagrangian

$$\widetilde{\mathcal{L}}_{Q_{\scriptscriptstyle BRS}} \equiv \widetilde{\mathcal{L}} + \alpha(t)Q_{\scriptscriptstyle BRS} + \overline{\alpha}(t)\overline{Q}_{\scriptscriptstyle BRS} \tag{82}$$

We shall hang around here for a moment spending some time on the lagrangian $\widetilde{\mathcal{L}}_{Q_{\scriptscriptstyle BRS}}$ above because it allows us to do some crucial observations on the counting of the degrees of freedom. As we said in the previous section the lagrangian $\widetilde{\mathcal{L}}_{susy}$ of eq.(47) has fewer degrees of freedom than the standard one $\widetilde{\mathcal{L}}$ of eq.(7), and the same happens with $\widetilde{\mathcal{L}}_{Q_{\scriptscriptstyle BRS}}$. In fact in $\widetilde{\mathcal{L}}_{Q_{\scriptscriptstyle BRS}}$ we have two primary constraints:

$$\Pi_{\alpha} = 0 \quad ; \quad \Pi_{\overline{\alpha}} = 0 \tag{83}$$

which generate two secondary ones:

$$Q_{\scriptscriptstyle BRS} = 0 \quad ; \quad \overline{Q}_{\scriptscriptstyle BRS} = 0 \tag{84}$$

All these four constraints are first class so we need four gauge-fixings. The total counting [11] is then $8n$ (original variables) $+4$ (gauge variables and momenta) -4 (constraints) -4 (gauge-fixings) for a total of $8n-4$ phase-space variables.

At this point one question that arises naturally is: "*Is it possible to have a lagrangian with the local invariances generated by Q_{BRS} and $\overline{Q}_{BRS}$, but with the same number of degrees of freedom as $\widetilde{\mathcal{L}}$?*". The answer is yes. In fact let us start from the following lagrangian:

$$\widetilde{\mathcal{L}}'_{Q_{BRS}} \equiv \widetilde{\mathcal{L}} - \dot{\alpha}(t)Q_{BRS} - \dot{\overline{\alpha}}(t)\overline{Q}_{BRS} \tag{85}$$

It is easy to check that it is invariant under the following set of local transformations generated by the Q_{BRS} and $\overline{Q}_{BRS}$:

$$\begin{aligned} \delta(\cdot) &= [\epsilon(t)Q_{BRS} + \overline{\epsilon}(t)\overline{Q}_{BRS}, (\cdot)] \\ \delta\alpha &= -i\epsilon \\ \delta\overline{\alpha} &= -i\overline{\epsilon} \end{aligned} \tag{86}$$

where we have indicated with $(\cdot)$ any of the variables $(\varphi^a, c^a, \lambda_a, \overline{c}_a)$. Anyhow at the same time it is easy to check that the lagrangian $\widetilde{\mathcal{L}}'_{Q_{BRS}}$ of (85) has only two primary constraints

$$\begin{aligned} \Pi_\alpha &= Q_{BRS} \\ \Pi_{\overline{\alpha}} &= \overline{Q}_{BRS} \end{aligned} \tag{87}$$

and no secondary ones. The above two constraints are first class so we need just two gauge-fixings and not four as before. The counting of degrees of freedom now goes as follows: $8n$ (original variables)$+4$(gauge variables and momenta)-2 (constraints)-2 (gauge fixings) for a total of $8n$ variables. So we see that the system described by the lagrangian $\widetilde{\mathcal{L}}'_{Q_{BRS}}$ of eq.(85) has the same number of degrees of freedom as the original $\widetilde{\mathcal{L}}$. In appendix E we will show how the constraints (87) act in the Hilbert space of the system. The fact that $\widetilde{\mathcal{L}}'_{Q_{BRS}}$ and $\widetilde{\mathcal{L}}$ are somehow equivalent could also be understood by doing an integration by part of the terms of eq.(85) containing $\dot{\alpha}$ and $\dot{\overline{\alpha}}$. The integration by parts produces, with respect to $\widetilde{\mathcal{L}}$, some terms which vanish because of the conservation of Q_{BRS} and $\overline{Q}_{BRS}$.

Even for the invariance under local susy of the previous section a mechanism like the one above could be implemented. One just needs to add to $\widetilde{\mathcal{L}}$ terms like those in eq.(47) but with the gauge fields replaced by their derivatives. This we feel may have been what has happened in ref. [21] where the authors have a system with local susy which has anyhow the same effective number of degrees of freedom as the model with global susy.

Going back to the lagrangian (85), we should mention that it is not the first time people thought of making the BRS-antiBRS invariance local [22]. We will not expand this issue here because we want to stick to the susy symmetry. We will come back in ref. [18] to the issue of gauging the BRS symmetry and all the Isp(2) charges of eqs.(20)-(24).

Let us now return to eq.(78). The two choices which led to eq.(81) and (82) are not the only ones consistent with the transformations (80). Another choice is

$$\left\{\begin{array}{l}\alpha=-\overline{\beta}\\ \overline{\alpha}=\beta\end{array}\right. \quad \left\{\begin{array}{l}\epsilon=-\overline{\eta}\\ \overline{\epsilon}=\eta\end{array}\right. . \tag{88}$$

With this choice the lagrangian that we get from (78) is:

$$\widetilde{\mathcal{L}}_{eq.}\equiv\widetilde{\mathcal{L}}+\alpha(t)Q_{(1)}+\overline{\alpha}(t)Q_{(2)}+g(t)\widetilde{\mathcal{H}} \tag{89}$$

The suffix (*eq.*) on the lagrangian is for "equivariant" and the reason will be clear later on. The $Q_{(1)}, Q_{(2)}$ on the RHS of eq. (89) are

$$\left\{\begin{array}{l}Q_{(1)}\equiv Q_{BRS}-\overline{N}_{H}\\ Q_{(2)}\equiv\overline{Q}_{BRS}+N_{H}\end{array}\right. \tag{90}$$

Note that these charges, with respect to the Q_H and $\overline{Q}_H$ of eq.(40), are somehow twisted in the sense that here we sum the Q_{BRS} with $\overline{N}$ and not with N and viceversa for the $\overline{Q}_{BRS}$. It is easy to check that:

$$Q_{(1)}^2=Q_{(2)}^2=-i\widetilde{\mathcal{H}} \quad ; \quad [Q_{(1)},Q_{(2)}]=0 \tag{91}$$

So these two charges generate two supersymmetry transformations which are anyhow different from those generated by the supersymmetry generators of eq.(40). The lagrangian of eq.(89) has two *local* susy invariances but different from the ones of $\widetilde{\mathcal{L}}_{susy}$ (47). In order to get the lagrangian (47) we should have made in eq.(78) and (80) the following choice:

$$\left\{\begin{array}{l}\alpha=-\beta=\overline{\psi}\\ \overline{\alpha}=\overline{\beta}=\psi\end{array}\right. \quad \left\{\begin{array}{l}\epsilon=-\eta\\ \overline{\epsilon}=\overline{\eta}.\end{array}\right. \tag{92}$$

Going back to eq.(89), let us now restrict the lagrangian to the following one:

$$\widetilde{\mathcal{L}}_{eq.}=\widetilde{\mathcal{L}}+\alpha(t)Q_{(1)}+g(t)\widetilde{\mathcal{H}} \tag{93}$$

which is locally invariant only under one susy and the symmetry transformations are:

$$\begin{cases} \delta(\cdot) = [\epsilon Q_{(1)} + \tau\widetilde{\mathcal{H}}, (\cdot)] \\ \delta\alpha = i\dot{\epsilon} \\ \delta g = 2i\alpha\epsilon + i\dot{\tau} \end{cases} \tag{94}$$

where $(\cdot)$ indicates any of the variables $(\varphi^a, c^a, \lambda_a, \overline{c}_a)$ and $\epsilon(t)$ and $\tau(t)$ are infinitesimal parameters. Because of this gauge invariance, we have to handle the system either via the Faddeev procedure [10] or the BFV method [11]. We will follow this last one. The constraints (primary and secondary) derived from (93) are:

$$\begin{cases} \Pi_\alpha = 0 \\ \Pi_g = 0 \\ Q_{(1)} = 0 \\ \widetilde{\mathcal{H}} = 0 \end{cases} \tag{95}$$

where Π_α and Π_g are respectively the momenta conjugate to the gauge fields $\alpha(t)$ and $g(t)$. The BFV procedure, as explained in the previous section, tells us to add four new ghosts and their respective momenta to the system. We will indicate them as follows:

$$\begin{cases} (C^{(1)}, C^H, \overline{C}_{(1)}, \overline{C}_H) \\ (\overline{\mathcal{P}}_{(1)}, \overline{\mathcal{P}}_H, \mathcal{P}_{(1)}, \mathcal{P}_H) \end{cases} \tag{96}$$

We shall impose the following graded-commutators:

$$\begin{cases} [g, \Pi_g] = [C^{(1)}, \overline{\mathcal{P}}_{(1)}] = [\overline{C}_{(1)}, \mathcal{P}_{(1)}] = 1 \\ [\alpha, \Pi_\alpha] = [C^H, \overline{\mathcal{P}}_H] = [\overline{C}_H, \mathcal{P}_H] = 1 \end{cases} \tag{97}$$

In the first line above the variables are all "bosonic" while in the second one are all Grassmannian. Equipped with all these tools we will now build the BFV-BRS [11] charge associated to our constraints:

$$\Omega_{BRS}^{(eq.)} \equiv C^{(1)}Q_{(1)} + C^H\widetilde{\mathcal{H}} + \mathcal{P}_{(1)}\Pi_\alpha + \mathcal{P}_{(H)}\Pi_g + i(C^{(1)})^2\overline{\mathcal{P}}_H \tag{98}$$

It is easy to check that $(\Omega_{BRS}^{(eq.)})^2 = 0$ as a BRS charge should be. The next step, analogous to what we did in section 3, is to select as physical states those annihilated by the $\Omega_{BRS}^{(eq).}$ charge:

$$\Omega_{BRS}^{(eq.)} \mid \text{phys}\rangle = 0 \tag{99}$$

Because of the nilpotent character of the $\Omega_{BRS}^{(eq.)}$, we should remember that two physical states are equivalent if they differ by a BRS variation:

$$\mid \text{phys-2}\rangle = \mid \text{phys-1}\rangle + \Omega_{BRS}^{(eq.)} \mid \chi\rangle \tag{100}$$

Performing the standard procedure [11] of abelianizing the constraints (95) and building the analog of the $\Omega^{(F)}_{BRS}$ of eq.(71), it is then easy to see that the physical state condition (99) is equivalent to the following four conditions:

$$\left\{\begin{array}{l} \widetilde{\mathcal{H}} \mid \text{phys}\rangle = 0 \\ Q_{(1)} \mid \text{phys}\rangle = 0 \\ \Pi_{\alpha} \mid \text{phys}\rangle = 0 \\ \Pi_{g} \mid \text{phys}\rangle = 0 \end{array}\right. \tag{101}$$

Let us now pause for a moment and, for completeness, let us briefly review the concept of equivariant cohomology (for references see [15]). Let us indicate with ψ and χ two *inhomogeneus* forms on a symplectic space and with V a vector field on the same space. One says that the form ψ is equivariantly closed but not exact with respect to the vector field V if the following conditions are satisfied:

$$\left\{\begin{array}{l} \mathcal{L}_{V}\psi = 0 \\ \mathcal{L}_{V}\chi = 0 \\ (d-\iota_{V})\psi = 0 \\ \psi \neq (d-\iota_{V})\chi \end{array}\right. \tag{102}$$

The forms ψ and χ have to be inhomogeneous because, while the exterior derivative d increases the degree of the form of one unit, the contraction with the vector field ι_V decreases it of one unit, so the third and fourth relations in eq.(102) would never have a solution if ψ and χ were homogeneous in the form degree. From now on let us use the notation: $d_{eq.} \equiv (d-\iota_V)$ and let us try to interpret the relations contained in eq.(102). Restricting the forms to satisfy the first two relations contained in (102) and noting that $d^2_{eq.} = -\mathcal{L}_V$, we have that on this restricted space $[d_{eq.}]^2 = 0$, and so $d_{eq.}$ acts as an exterior derivative. If we now consider the last two relations of (102) it is then clear that they define a cohomology problem for $d_{eq.}$.

There are other more abstract definitions of equivariant cohomology [13] based on the *basic* cohomology of the Weil algebra associated to a Lie-algebra, but we will not dwell on it here. Equivariant cohomology is a concept that entered also the famous localization formula of Duistermaat and Heckman [14] thanks to the work of Atiyah and Bott and into Topological Field Theory thanks to the work of R. Stora and collaborators [16].

Let us now go back to our lagrangian $\widetilde{\mathcal{L}}_{eq.}$ of eq. (93) whose physical state space is restricted by the conditions (101) because of the gauge invariance given in (94). It is easy to realize that the first two physical state conditions of (101) are equivalent to the first and third conditions of eq. (102) once the vector field V is identified with the Hamiltonian vector field $(dH)^{\sharp}$ [4]. In fact let us remember the correspondence described in section 2 between standard operations in differential geometry and in our

formalism and in particular formula (37). This tells us that the Lie-derivative $\mathcal{L}_{dH^\sharp}$ acts on a form as the commutator of $\widetilde{\mathcal{H}}$ with the same form written in terms of c^a variables. This commutator gives the same result as the action of $\widetilde{\mathcal{H}}$ on functions of φ^a and c^a once $\widetilde{\mathcal{H}}$ is written as a differential operator like in (101). So eq. (37) proves that the first condition in both eqs. (102) and (101) is the same:

$$\mathcal{L}_{(dH^\sharp)}\psi = 0 \longrightarrow \widetilde{\mathcal{H}} \mid \text{phys}\rangle = 0 \tag{103}$$

Next let us look at the second condition in (101) and the third in (102). From the form (90) of $Q_{(1)}$ we see that its first term, the Q_{BRS}, via the correspondence given by eq.(29), corresponds to the exterior derivative d which is exactly the first term contained in the third relation of eq.(102). The second term in $Q_{(1)}$ is the $\overline{N}$ which is given in eq.(39) and can be written as

$$\overline{N}_H = [\overline{Q}_{BRS}, H] \tag{104}$$

From the relation (34) we see that we can interpret $\overline{N}_H$ as the Hamiltonian vector field built out of the function H. Its action as a differential operator on forms is then given by eq.(30), that means it acts as the interior contraction, $\iota_{(dH^\sharp)}$, of the Hamiltonian vector field with forms. This basically proves the correspondence between the second relation of (101) and the third of (102):

$$(d - \iota_{(dH^\sharp)})\psi = 0 \longrightarrow Q_{(1)} \mid \text{phys}\rangle = 0 \tag{105}$$

Note that this correspondence would not hold if we had gauged the Q_H, like we did in section 3. In fact the Q_H, being made of Q_{BRS} and N_H and not $\overline{N}_H$, would not have had the meaning of equivariant exterior derivative.

Let us now conclude the proof that the conditions (101) are really equivalent to the equivariant cohomology problem given by eq. (102). We have already explained in eqs. (103) and (105) the correspondences:

$$\begin{gathered}\mathcal{L}_V\psi = 0 \longleftrightarrow \widetilde{\mathcal{H}} \mid \text{phys}\rangle = 0 \\ (d - \iota_V)\psi = 0 \longleftrightarrow Q_{(1)} \mid \text{phys}\rangle = 0.\end{gathered}$$

We haven't discussed yet the 2nd and 4th equations in (102). These conditions are equivalent to the following statement:

$$\{\psi = (d - \iota_V)\chi \quad \text{with} \quad \mathcal{L}_V\chi = 0\} \quad \Longrightarrow \quad \{\psi \simeq 0\}, \tag{106}$$

where the symbol $\simeq$ means "cohomologically equivalent ". Therefore, if we want to complete the proof of the correspondence between the $|\text{phys}\rangle$ states of (101) and the ψ of (102), we must show that:

$$\{|\text{phys}\rangle = Q_{(1)}|\chi\rangle \quad \text{with} \quad \widetilde{\mathcal{H}}|\chi\rangle = 0\} \quad \Longrightarrow \quad \{|\text{phys}\rangle \simeq 0\}. \tag{107}$$

Note that the state $|\chi\rangle$ is not required to be physical, but only to satisfy the LHS of eq. (107); this implies that $|\chi\rangle$ in general does not satisfy the third and the fourth conditions of (101). This means that $|\chi\rangle$ in general can depend on g and α. The point is that this dependence, due to the LHS of eq. (107) and to the requirement that $|\mathrm{phys}\rangle$ does not depend on g and α, must have the following form, as proven in Appendix F:

$$|\chi\rangle = |\chi_0\rangle + |\zeta;\alpha,g\rangle \tag{108}$$

where $|\chi_0\rangle$ is independent of both g and α, and $|\zeta;\alpha,g\rangle \in \ker Q_{(1)}$. Moreover if $|\chi\rangle \in \ker\widetilde{\mathcal{H}}$ (as imposed by eq. (107)), also $|\chi_0\rangle \in \ker\widetilde{\mathcal{H}}$ as one can check by applying $Q^2_{(1)} = -i\widetilde{\mathcal{H}}$ to both members of eq. (108). We are now ready to show that states of the form (107) are cohomologically equivalent to zero according to the cohomology defined by $\Omega^{(eq.)}_{BRS}$ of eq. (98). The proof goes as follows:

$$\begin{aligned}
|\,\mathrm{phys}\rangle &= Q_{(1)}\,|\,\chi\rangle \\
&= Q_{(1)}\,|\,\chi_0\rangle \\
&= [C^{(1)}]^{-1}[C^{(1)}Q_{(1)} + C^{H}\widetilde{\mathcal{H}} + \mathcal{P}_{(1)}\Pi_{\alpha} + \mathcal{P}_{H}\Pi_{g} + i(C^{(1)})^2\overline{\mathcal{P}}_{H}]\,|\,\chi_0\rangle \\
&= [C^{(1)}]^{-1}\Omega^{eq.}_{BRS}\,|\,\chi_0\rangle \\
&= \Omega^{(eq.)}_{BRS}\,|\,\chi'\rangle
\end{aligned} \tag{109}$$

(where we have defined $|\,\chi'\rangle \equiv [C^{(1)}]^{-1}\,|\,\chi_0\rangle$) and therefore $|\,\mathrm{phys}\rangle$ is cohomologically equivalent to zero. Note that the ghost $C^{(1)}$ is bosonic in character and so we can build its inverse. In the third equality of (109) we have used the fact that the second, the third and the fourth terms give zero when applied to $|\chi_0\rangle$. The last term $(i(C^{(1)})^2\overline{\mathcal{P}}_{H})$ also annihilates $|\chi_0\rangle$ by a similar reasoning based on the fact that $|\,\mathrm{phys}\rangle$ cannot contain any dependence on C^{H}. This concludes our proof.

Basically with our path-integral we have managed to get the propagation of equivariantly non-trivial states by properly gauging the susy. This is what the susy is telling us from a geometrical point of view. It is not the first time that the equivariant cohomology is reduced to a sort of BRS formalism [17], but differently from these authors the BRS-BFV charge we obtained is really linked to a local invariance problem associated to the lagrangian (93). We feel that this detailed analysis of the problem will be appreciated by the community of physicists not last for his pedagogical value.

At this point we can say that, by using the trick of gauging it, we have understood the geometry lying behind our susy. Most probably a further geometrical understanding of our original path-integral (6) will emerge once we gauge [18] also some of the other universal symmetries found previously [1] [23]. These are made of the generators of an Isp(2) superalgebra and of a noncanonical charge [23] whose role is to rescale the entire action of any classical system. To better understand the geometry lying behind them we plan to study the effect of these local symmetries on the superspace which is made [1] [23] of the time t and of two Grassmannian partners of the time which we called $\theta, \overline{\theta}$.

We feel that this attempt to better understand the geometry lying behind our functional formulation of CM will help not only in unveiling some of the unsolved problems of Classical Mechanics (ergodicity and integrability being some of them) but also in throwing light on the transition to the quantum regime which, one-hundred years after the invention of $\hbar$, is still an issue in need of a deeper undestanding. The reader may wonder what the quantization has to do with the geometry lying behind our classical path-integral and its associated superspace $(t, \theta, \bar{\theta})$. The explanation can be found in ref. [24] where it was shown that the process of quantization is equivalent to freezing to zero the Grassmannian partners of time $(\theta, \bar{\theta})$. This freezing to zero must be hiding some deep geometrical structures which we have not completely understood yet. The same may happen for the new universal symmetry that we discovered in ref. [23]. This is a symmetry which rescales the entire action of any classical system and it is clearly broken at the quantum level by the presence of $\hbar$. We will return to these problems in other papers.

In this paper we will continue to concentrate on Classical Mechanics and try to put the geometrical basis to attack in our formalism issues like integrability or ergodicity. This is what we plan to do in the next section.

5 Motion On Constant-Energy Surfaces.

In the previous sections we have gauged the susy and we have ended up with a constrained motion on the hypersurfaces given by eq.(61) or (95). In this section we shall reverse the procedure. We want to constrain the motion on some particular hypersurfaces of phase-space and see which local symmetries the associated lagrangian will exhibit. The hypersurfaces we choose are those defined by fixed values of the constants of motion. We will explain later the reasons for this choice.

Let us start with the constant energy surface: $H(p, q) = E$. The most natural thing to do is to add this constraint to the $\widetilde{\mathcal{L}}$ of eq.(7):

$$\widetilde{\mathcal{L}}_{E} \equiv \widetilde{\mathcal{L}} + f(t)(H - E) \tag{110}$$

where $f(t)$ is a gauge variable. This lagrangian has the following local invariance :

$$\begin{cases} \delta_{H}(\cdot) = [\tau(t)H, (\cdot)] \\ \delta_{H} f(t) = i\dot{\tau}(t) \end{cases} \tag{111}$$

where $\tau(t)$ is an infinitesimal bosonic parameter and we have indicated with $(\cdot)$ any of the variables $(\varphi^{a}, c^{a}, \lambda_{a}, \overline{c}_{a})$. Anyhow this is not the whole story. In fact if we restrict the original phase-space to be a constant energy surface, the forms c^{a} themselves must

be restricted to be those living only on the energy surface, that means they must be "perpendicular" to the gradient of the Hamiltonian. This constraint is:

$$c^a\partial_a H = 0 = N_{\scriptscriptstyle H} \tag{112}$$

So basically we have to impose that the $N_{\scriptscriptstyle H}$-function of eq. (39) be zero. This is a further constraint we should add to the $\widetilde{\mathcal{L}}_{\scriptscriptstyle E}$ of eq.(110). One may think that an analogous restriction has to be done also for the vector fields considering that forms and tensor fields are paired by the symplectic matrix as explained in eqs.(32)(33). If we accept this we will have to add the condition that the $\overline{N}_{\scriptscriptstyle H}$ of eq.(39) be zero.

A manner to get all these constraints automatically, without having to add them by hand, is beautifully achieved if we request that the new Hamiltonian $\widetilde{\mathcal{H}}_{\scriptscriptstyle E}$, which describes the motion on the energy surface, be a Lie-derivative of an Hamiltonian flow like the original $\widetilde{\mathcal{H}}$ was. $\widetilde{\mathcal{H}}_{\scriptscriptstyle E}$ must be a Lie-derivative because, after all, the motion is the same as before. This time the difference is that we fix a particular value of the energy and so we include this initial condition directly into the lagrangian. If $\widetilde{\mathcal{H}}_{\scriptscriptstyle E}$ is a Lie-derivative of an Hamiltonian flow then, from what we said below eq.(38), we gather that $\widetilde{\mathcal{H}}_{\scriptscriptstyle E}$ must be of the following form:

$$\widetilde{\mathcal{H}}_{\scriptscriptstyle E} = [Q^{\scriptscriptstyle E}_{\scriptscriptstyle BRS}[\overline{Q}^{\scriptscriptstyle E}_{\scriptscriptstyle BRS}, (\cdot)]] \tag{113}$$

that means it must be the BRS variation of the antiBRS variation of some function that we indicate in (113) with $(\cdot)$. The BRS and antiBRS charges in (113) are not the ones of $\widetilde{\mathcal{H}}$, that means those of eqs.(20)(21). For this reason we have indicated them with different symbols.

If $\widetilde{\mathcal{H}}_{\scriptscriptstyle E}$ is of the form above then the associated lagrangian must be BRS invariant. Let us start from the $\widetilde{\mathcal{L}}_{\scriptscriptstyle E}$ in eq.(110) and see if it is BRS invariant at least under the old $Q_{\scriptscriptstyle BRS}$. It is easy to do that calculation and we get:

$$[Q_{\scriptscriptstyle BRS}, \widetilde{\mathcal{L}} + f(H-E)] = f(t)N_{\scriptscriptstyle H} \neq 0 \tag{114}$$

So it is not BRS invariant. The way out is to add to $\widetilde{\mathcal{L}}_{\scriptscriptstyle E}$ the $N_{\scriptscriptstyle H}$ multiplied by a gauge field. The new lagrangian is:

$$\widetilde{\mathcal{L}}^{\prime}_{\scriptscriptstyle E} \equiv \widetilde{\mathcal{L}} + f(t)(H-E) + i\alpha(t)N_{\scriptscriptstyle H} \tag{115}$$

where $\alpha(t)$ is the Grassmannian gauge field. This lagrangian is BRS-invariant provided that we define a proper BRS-variation also on the gauge-fields $\alpha(t)$ and $f(t)$. These proper BRS transformations are:

$$\left\{\begin{array}{l}\delta(\cdot)=[\epsilon Q_{BRS},(\cdot)]\\ \delta\alpha=i\epsilon f\\ \delta f=0\end{array}\right. \tag{116}$$

where we have indicated with $(\cdot)$ any of the $(\varphi^a, c^a, \lambda_a, \overline{c}_a)$ and with Q_{BRS} the old BRS charge of eq.(20).

Next let us notice that if the form of $\widetilde{\mathcal{H}}_E$ is the one of eq. (113) then the associated lagrangian has to be also antiBRS-invariant. Let us check if this happens with the $\widetilde{\mathcal{L}}'_E$ of eq.(115):

$$[\overline{Q}_{BRS}, \widetilde{\mathcal{L}}'_E] = f(t)\overline{N}_H - \alpha(t)\widetilde{\mathcal{H}} \tag{117}$$

So it is not antiBRS invariant and the way out is again to add to $\widetilde{\mathcal{L}}'_E$ the generators appearing on the RHS of (117). The final lagrangian is:

$$\widetilde{\mathcal{L}}''_E \equiv \widetilde{\mathcal{L}} + f(t)(H-E) + i\alpha(t)N_H + i\overline{\alpha}(t)\overline{N}_H - g(t)\widetilde{\mathcal{H}} \tag{118}$$

where $(f, \alpha, \overline{\alpha}, g)$ are gauge-fields. So we see that the request that our $\widetilde{\mathcal{H}}_E$ be a Lie-derivative (113) has automatically produced the constraints $N_H = 0$ and $\overline{N}_H = 0$ that otherwise we would have had to add by hand like we did in the reasoning leading to eq.(112).

The lagrangian $\widetilde{\mathcal{L}}''_E$ is invariant under the following generalized BRS and antiBRS transformations:

$$\delta_{Q_{BRS}} \equiv \left\{\begin{array}{l}\delta(\cdot)=[\epsilon Q_{BRS},(\cdot)]\\ \delta\alpha=i\epsilon f\\ \delta\overline{\alpha}=0\\ \delta f=0\\ \delta g=\epsilon\overline{\alpha}\end{array}\right. \qquad \overline{\delta}_{\overline{Q}_{BRS}} \equiv \left\{\begin{array}{l}\overline{\delta}(\cdot)=[\overline{\epsilon}\overline{Q}_{BRS},(\cdot)]\\ \overline{\delta}\alpha=0\\ \overline{\delta}\overline{\alpha}=i\overline{\epsilon}f\\ \overline{\delta}f=0\\ \overline{\delta}g=-\overline{\epsilon}\alpha\end{array}\right. \tag{119}$$

It is straightforward to build the BRS-antiBRS charges which produce the variations indicated above. They are:

$$Q^E_{BRS} \equiv Q_{BRS} + if\Pi_\alpha + i\overline{\alpha}\Pi_g \tag{120}$$
$$\overline{Q}^E_{BRS} \equiv \overline{Q}_{BRS} + if\Pi_{\overline{\alpha}} - i\alpha\Pi_g \tag{121}$$

where Π_α, $\Pi_{\overline{\alpha}}$, Π_g are the momenta conjugate to the variables $\alpha, \overline{\alpha}, g$ and their graded commutators are:

$$[\alpha, \Pi_\alpha] = [\overline{\alpha}, \Pi_{\overline{\alpha}}] = i[\Pi_g, g] = i[f, \Pi_f] = 1 \tag{122}$$

The new BRS and antiBRS charges are nilpotent, as BRS charges should be, and anticommute among themselves

$$(Q^E_{BRS})^2 = (\overline{Q}^E_{BRS})^2 = [Q^E_{BRS}, \overline{Q}^E_{BRS}] = 0 \tag{123}$$

Having obtained these charges it is then easy to prove that the $\widetilde{\mathcal{H}}''_E$ associated to the $\widetilde{\mathcal{L}}''_E$ of eq.(118) has the form (113) with the $(\cdot)$ in (118) given by $-i(H+g(H-E))$, i.e:

$$\widetilde{\mathcal{H}}''_E = -i[Q^E_{BRS}[\overline{Q}^E_{BRS}, H + g(H-E)] \tag{124}$$

This shows, with respect to the $\widetilde{\mathcal{H}}$ of eq.(38), that the 0-form out of which the Hamiltonian vector field is built is not H but $H + g(H-E)$. This is natural in the sense that this 0-form feels the constraint $H - E = 0$.

The symplectic structure behind our construction can be made more manifest if we introduce the following notation:

$$\left\{ \begin{array}{l} \varphi^A \equiv (\varphi^a; g, \Pi_f) \\ \lambda_A \equiv (\lambda_a; \Pi_g, f) \\ c^A \equiv (c^a; \overline{\alpha}, \Pi_\alpha) \\ \overline{c}_A \equiv (\overline{c}_a; \Pi_{\overline{\alpha}}, \alpha) \end{array} \right. \tag{125}$$

where the index in capital letter $(\cdot)^A$ runs from 1 to $2n+2$ while the one in small letters $(\cdot)^a$ runs from 1 to $2n$ and it refers to the usual variables $(\varphi^a, c^a, \lambda_a, \overline{c}_a)$. Let us also introduce an enlarged symplectic matrix:

$$\omega^{AB} = \begin{pmatrix} \omega^{ab} & 0 \\ 0 & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix} \tag{126}$$

and then, using the definitions (125) and (126), the BRS-antiBRS charges (120)(121) can be written in the following compact form:

$$\left\{ \begin{array}{l} Q^E_{BRS} = ic^A\lambda_A \\ \overline{Q}^E_{BRS} = i\overline{c}_A\omega^{AB}\lambda_B. \end{array} \right. \tag{127}$$

Note that this form resembles very much the one of the original BRS and antiBRS charges (20)(21). It is also straightforward to prove that the $\widetilde{\mathcal{H}}''_E$ has an N=2 supersymmetry like the old $\widetilde{\mathcal{H}}$. To build the susy charges we should first construct the $(N_H, \overline{N}_H)$ charges analogous to those in (39). Replacing in (39) the symplectic matrix and the variables with those constructed respectively in (126) and (125), and the 0-form H with the 0-form $H + g(H-E)$ entering the $\widetilde{\mathcal{H}}_E$, we get:

$$\begin{cases} N_H^E = c^A \partial_A (H + g(H - E)) \\ \overline{N}_H^E = \overline{c}_A \omega^{AB} \partial_B (H + g(H - E)) \end{cases} \tag{128}$$

The supersymmetry charges analogous to those in eq.(40) are then

$$\begin{cases} Q_H^E \equiv Q_{BRS}^E - \beta N_H^E \\ \overline{Q}_H^E \equiv \overline{Q}_{BRS}^E + \beta \overline{N}_H^E \end{cases} \tag{129}$$

where β is a dimensional parameter like the one appearing in (40). It is then easy to check that:

$$[Q_H^E, \overline{Q}_H^E] = 2i\beta \widetilde{\mathcal{H}}_E'' \tag{130}$$

Up to now we have found which are the global symmetries of our lagrangian (118), but let us not forget that the goal of this section was to find out if, by imposing a constraint from outside like the one of being on a constant energy surface, we would get a lagrangian with local symmetries. It is actually so and a first hint was given by the local symmetry of eq.(111). The full set of local invariances of the lagrangian $\widetilde{\mathcal{L}}_E''$ of eq.(118) is:

$$\begin{cases} \delta(\cdot) = [\tau H + \bar{\eta} \overline{N}_H + \eta N_H + \epsilon \widetilde{\mathcal{H}}, (\cdot)] \\ \delta f = i\dot{\tau} \\ \delta \alpha = \dot{\eta} \\ \delta \bar{\alpha} = \dot{\bar{\eta}} \\ \delta g = -i\dot{\epsilon} \end{cases} \tag{131}$$

where $(\tau, \eta, \bar{\eta}, \epsilon)$ are the local gauge-parameters depending on t, and with $(\cdot)$ we have indicated the variables $(\varphi^a, c^a, \lambda_a, \overline{c}_a)$.

The above local symmetry is not a local supersymmetry as in the previous sections but a different graded one whose generators are $(H, N, \overline{N}, \widetilde{\mathcal{H}})$. While before, in section 3 and 4, the local symmetry was a clearly recognizable one but the constraints — being in the enlarged space $(\varphi^a, c^a, \lambda_a, \overline{c}_a)$ — were hard to visualize, here we have the inverse situation: the constraint (the constant energy one) is easy to visualize but not so much the local symmetries.

For a moment let us stop these formal considerations and let us check that the Hamiltonian in eq. (124) is the correct one. The procedure we have followed here of constraining the motion on a constant energy surface can be applied also to any other constant of motion $I(\varphi)$. The result would be the following Hamiltonian:

$$\widetilde{\mathcal{H}}_I'' = \widetilde{\mathcal{H}} - f[I(\varphi) - k] - i\bar{\alpha}\overline{N}_{(I)} - i\alpha N_{(I)} + g\widetilde{\mathcal{I}} \tag{132}$$

where k is a constant and

$$
\begin{cases} N_{(I)} = c^a \partial_a I \\ \overline{N}_{(I)} = \overline{c}_a \omega^{ab} \partial_b (I) \\ \widetilde{\mathcal{I}} = -i[Q_{BRS}, [\overline{Q}_{BRS}, I(\varphi)]] \end{cases} \tag{133}
$$

If we had an integrable system with n constants of motion I_i in involution we would get as Hamiltonian the following one:

$$
\widetilde{\mathcal{H}}''_{int.} = \widetilde{\mathcal{H}} - \sum_i \{f_i [I_i(\varphi) - k_i] - i\bar{\alpha}_i \overline{N}_{(I)_i} - i\alpha_i N_{(I)_i} + g\widetilde{\mathcal{I}}_i\} \tag{134}
$$

Let us now do a counting of the effective degrees of freedom of the Hamiltonian $\widetilde{\mathcal{H}}''_I$ of eq.(132). We have $8n$ variables $(\varphi^a, c^a, \lambda_a, \overline{c}_a)$, plus 4 gauge fields $(f, \alpha, \bar{\alpha}, g)$, plus 4 momenta associated to these gauge fields minus 4 primary constraints (which are the gauge-momenta equal zero), minus 4 secondary constraints ($I - k = 0$, $N_{(I)} = 0$, $\overline{N}_{(I)} = 0$, $\widetilde{\mathcal{I}} = 0$)) minus 8 gauge-fixings for a total of $8n - 8$ independent phase-space variables. For the Hamiltonian of an integrable system like $\widetilde{\mathcal{H}}''_{int.}$ this counting would give $8n - 8n = 0$ as effective number of phase-space variables describing the system. *This is absurd !* This situation could already be seen in the one-dimensional harmonic oscillator where $n = 1$ and we have just one constant of motion (the energy). The number of variables of the associated $\widetilde{\mathcal{H}}''_E$ would be $8n - 8 = 8 - 8 = 0$. One could claim that our $\widetilde{\mathcal{H}}''_{int.}$, having zero degrees of freedom, actually describes a Topological-Field-Theory model, and maybe it is so but for sure it does not describe the motion taking place on the tori of an integrable system. On the tori we have the angles which vary with time but here, having effectively zero phase-space variables, we do not have any motion taking place at all. If it is a topological theory at most the $\widetilde{\mathcal{H}}''_{int.}$ can describe some static *geometric* feature of our system. This in itself would be interesting and that is why we have carried this construction so far. We hope to come back to this issue in future papers but for the moment we want to go back from where we started, that is eq.(110) and see which is the way to get an Hamiltonian describing really the motion on the constant energy surface.

What we basically want to get is an Hamiltonian whose counting of degrees of freedom is correct. At the basic phase-space level labelled by the variables φ we have $2n$ variables minus 1 constraint that is $H - E = 0$ so the total number is $2n - 1$. Going up to the space $(\varphi^a, c^a, \lambda_a, \overline{c}_a)$ this number should be multiplied by 4 that is $8n - 4$.

What went wrong in the construction of $\widetilde{\mathcal{L}}''_E$ of eq.(118)? One thing that we requested, but which was not necessary, was that the vector fields obey a constraint $\overline{N} = 0$ analogous to the one of the forms $N = 0$. We made that request only in order to maintain the standard pairing between tensor fields and forms which appear in any symplectic theory, but our theory is not a symplectic one anymore because the basic space in φ^a has odd dimension $2n - 1$ and cannot be a symplectic space. So let us

release the request of having $\overline{N} = 0$. We could have a weaker request by adding this constraint via the derivative of a lagrange multiplier (or gauge-field) in the same manner as we did in eq.(85). There we realized that adding constraints in this manner does not decrease the number of degrees of freedom. By consistency then also the $\widetilde{\mathcal{H}}$ constraint, which appeared together with the $\overline{N}$ via the eq.(117), should be added via the derivative of its associated lagrange multiplier. So in order to describe the motion on constant energy surfaces, instead of (118) the lagrangian we propose is:

$$L_{E} = \widetilde{\mathcal{L}} + f(H-E) + i\dot{\overline{\alpha}}\overline{N} + i\alpha N - \dot{g}\widetilde{\mathcal{H}} \tag{135}$$

The constraints (primary and secondary) are:

$$\begin{cases} \Pi_{f} = 0\ ; \quad H-E=0\ ; \\ \Pi_{\alpha} = 0\ ; \quad N = 0\ ; \\ \Pi_{\overline{\alpha}} = -i\overline{N}\ ; \\ \Pi_{g} = -\widetilde{\mathcal{H}}. \end{cases} \tag{136}$$

They are 6, all first class, and we need 6 gauge-fixings. So doing now the counting of independent variables in phase-space we have: $8n$ variables $(\varphi^{a}, c^{a}, \lambda_{a}, \overline{c}_{a})$, plus $4+4$ gauge-fields and their momenta, minus 6 constraints, minus 6 gauge fixings for a total of $8n-4$ which is exactly the number we wanted!

Let us analyze the difference between the last constraint in eq.(136) (that is $\Pi_{g} = -\widetilde{\mathcal{H}}$) and the one associated to the lagrangian $\widetilde{\mathcal{L}}''_{E}$ of eq. (118) (that is $\widetilde{\mathcal{H}} = 0$). This last constraint seems to totally freeze the motion while the one in eq.(136) does not freeze it but just foliates the space of values of $\widetilde{\mathcal{H}}$. Similar things can be said for the constraint $\overline{N} = 0$ associated to $\widetilde{\mathcal{L}}''_{E}$ and the one, $\Pi_{\alpha} = i\overline{N}$, associated to L_{E}. This last one would not force the vector fields in a configuration symplectically equivalent to the one of forms.

Let us now proceed to further analyze the lagrangian L_{E} of eq.(135). The associated Hamiltonian is:

$$H_{E} = \widetilde{\mathcal{H}} + \Pi_{\alpha}\dot{\alpha} + \Pi_{g}\dot{g} + \Pi_{f}\dot{f} + \Pi_{\overline{\alpha}}\dot{\overline{\alpha}} - f(H-E) - i\alpha N - i\dot{\overline{\alpha}}\overline{N} + \dot{g}\widetilde{\mathcal{H}} \tag{137}$$

where we had to leave in some velocities because we could not perform the Legendre transformation. From the above Hamiltonian we can go to the canonical one [9] by imposing the primary constraints. The result is:

$$H_{E}^{can.} \equiv \widetilde{\mathcal{H}} - f(H-E) - i\alpha N \tag{138}$$

It is easy to prove that this $H_{E}^{can.}$ is a Lie-derivative of a vector field but not of an Hamiltonian vector-field. To show that let us first define the following new variables:

$$
\begin{cases}
\varphi^A = (\varphi^a, \pi_f) \\
\lambda_A = (\lambda_a, f) \\
c^A = (c^a, \Pi_\alpha) \\
\overline{c}_A = (\overline{c}_a, \alpha)
\end{cases}
\tag{139}
$$

In this enlarged phase-space the BRS charge (or exterior derivative) is

$$
Q_{BRS}^{can.} = Q_{BRS} + if\Pi_\alpha \tag{140}
$$

and the analog of the Hamiltonian vector field $\overline{N}_H$ is

$$
\overline{N}_H^{can.} = \overline{N}_H - \overline{\alpha}(H - E) \tag{141}
$$

which is not an Hamiltonian vector field anymore because it cannot be written as the antiBRS variation of something as a Hamiltonian vector field should be (see eq.(34)).

The proof that $H_E^{can.}$ of eq.(138) is the Lie-derivative [4] of the vector field $\overline{N}_H^{can.}$ above comes from the fact that it can be written as the commutator of that vector field with the exterior derivative $Q_{BRS}^{can.}$ above:

$$
H_E^{can.} = -i[Q_{BRS}^{can.}, \overline{N}_H^{can.}] \tag{142}
$$

To prove this relation is straightforward. One just needs to use the standard commutators plus the following ones:

$$
[\alpha, \Pi_\alpha] = 1 \quad ; \quad [f, \Pi_f] = -i \tag{143}
$$

Eq.(142) implies that $H_E^{can.}$ of eq.(138) is invariant under the global BRS transformations generated by the $Q_{BRS}^{can.}$ of eq.(140). It is also easy to see that the lagrangian L_E of eq.(135) has the following local invariances different from those of eq.(131):

$$
\begin{cases}
\delta(\cdot) = [\tau H + \bar{\eta}\overline{N}_H + \eta N_H + \epsilon\widetilde{\mathcal{H}}, (\cdot)] \\
\delta f = i\dot{\tau} \\
\delta\alpha = \dot{\eta} \\
\delta\bar{\alpha} = \dot{\bar{\eta}} \\
\delta g = -i\epsilon
\end{cases}
\tag{144}
$$

Again, as before, this is a local symmetry but not a local supersymmetry.

Regarding the supersymmetry we can find a global one under which our $H_E^{can.}$ of eq.(138) is invariant. It is the one generated by the following charge:

$$
Q_{susy} = Q_{BRS}^{can.} + \overline{N}_H^{can.} \tag{145}
$$

which is a susy charge because it is easy to prove that:

$$[Q_{susy}]^2 = iH_E^{can.} \tag{146}$$

Differently from the $\widetilde{\mathcal{H}}$ of our original system, we do not have an N=2 supersymmetry like in eq.(91), but only an N=1 susy. This is due to the loss of a symplectic structure on the constant energy surface.

The reason to work out this supersymmetry is not just academical. In fact we proved in ref. [5] that there is a nice interplay between the loss of ergodicity of the system whose Hamiltonian is H and the spontaneous breaking of the susy of $\widetilde{\mathcal{H}}$. We proved in particular that if the susy of $\widetilde{\mathcal{H}}$ is unbroken then the system described by H is in the ergodic phase and that if the system is in the ordered phase (non-ergodic) then the susy of $\widetilde{\mathcal{H}}$ must be broken. We could not prove the inverse of these two statements that is that if the system is in the ergodic phase then the susy must be unbroken and that if the susy is broken then the system must be in an ordered or non-ergodic regime. The reason we could not prove these inverse statements was that the energy at which the motion took place had not been specified. We have no time here to explain the detailed reasons why this lack of specification could not allow us to do the inverse of that statement and we advice the reader interested in understanding this point to study in detail the full set of papers contained in ref. [5]. The ergodicity [6] is a concept which is strongly energy dependent: a system can be ergodic at some energy and not ergodic at other energies. So it was crucial to develop a formalism giving us the motion on constant energy surfaces like we have done here. The parameter E entering our $H_E^{can.}$ is not a phase-space variable and we can consider it as a coupling constant. We know that at some values of the coupling a symmetry can be broken while at others it can be restored. In (135) the term containing the energy is like a tadpole term because it is proportional to a term linear in the field (the field in this case is $f(t)$ while the coupling is E).

The attempt to have a formulation of the CPI in which E enters explicitly was tried before [25] but along a different route. In that paper E was not a coupling constant but a degree of freedom conjugate to time in a formulation of CM invariant under time-reparametrization. We think that, in order to understand the interplay *susy/ergodicity*, it is better to treat E as a coupling constant.

The next step would be to check whether the susy charge (145) we have in $H_E^{can.}$ is that for which the theorem [5] mentioned above, regarding the interplay *susy/ergodicity*, holds also in the inverse form. If this were the case then we would have a criterion to check whether a system (at some energy) is ergodic or not using a universal symmetry like susy. Maybe even a sort of Witten index could be built which, by signaling whether the susy is broken or not, could tell us whether the system is ergodic or not.

All this work will be left to future papers [18]. Here we wanted to stick to geometrical issues. We say “future papers” because there are several other difficulties that have to be cleared before really embarking on a full understanding of the interplay between susy and ergodicity. The main difficulty is the presence of zero and negative norm states which prevents the proper use of something like a Witten index for the study of the above mentioned interplay.

6 Conclusions

In this paper we have continued the study of the geometry lying behind a functional approach to CM developed in ref. [1]. In particular here we have focused our attention on a universal supersymmetry [5] which seemed to have a role both at the geometrical level, for its relation to the issue of equivariant cohomology, and at the dinamical level for its interplay with the concept of ergodicity [6].

We have clarified the first connection by making the susy local and building the BFV charge associated to this local symmetry. This has been done in great detail in order to better understand several issues present in the literature.

We have also put the geometrical basis to better understand the second connection, that is the interplay between susy and ergodicity. We have done it by formulating our system on constant energy surfaces and by thoroughly exploring the geometry of this formulation. We have brought to light both the geometrical meaning of the associated Hamiltonian and also the surviving global and local symmetries. We hope now to have in our hands all the weapons needed for the final attack on this issue of the interplay *susy/ergodicity*.

Appendices

A Appendix

In deriving eqs. (45) and (46), or even in checking the global symmetry under $Q_{\scriptscriptstyle H}$, we had to work out things involving the variation of the kinetic piece of $\widetilde{\mathcal{L}}$, i.e.:

$$[\epsilon Q_{\scriptscriptstyle H}, \lambda_a\dot{\varphi}^a + i\overline{c}_a\dot{c}^a] = (\delta\lambda_a)\dot{\varphi}^a + \lambda_a\frac{d}{dt}(\delta\varphi^a) + i(\delta\overline{c}_a)\dot{c}^a + i\overline{c}_a\frac{d}{dt}(\delta c^a) \tag{A.1}$$

In this step we have interchanged the variation "δ" with the time derivative $\frac{d}{dt}$. If we actually do the time derivative of a variation (for example of φ^a), we get

$$\begin{aligned}\frac{d}{dt}(\delta\varphi^a) \quad &= \tfrac{d}{dt}[\epsilon Q_{\scriptscriptstyle H}, \varphi^a] \\ &= [\tfrac{d}{dt}(\epsilon Q_{\scriptscriptstyle H}), \varphi^a] + [\epsilon Q_{\scriptscriptstyle H}, \tfrac{d\varphi^a}{dt}] \\ &= [\epsilon Q_{\scriptscriptstyle H}, \tfrac{d\varphi^a}{dt}] \\ &= \delta\left(\frac{d\varphi^a}{dt}\right),\end{aligned} \tag{A.2}$$

and if ϵ is a global parameter the third equality in the equation above holds (and as a result we can interchange the variation with the time derivative) only if we use the fact that $\frac{dQ_H}{dt} = 0$. We have supposed the same thing in the case of the local variations (45)(46), and the only extra term appearing with respect to eq.(A.2) is the one containing the $\dot{\epsilon}$. Using the conservation of $Q_{\scriptscriptstyle H}$ means that we have assumed that the equations of motion hold. Actually it is better not to assume that. In fact, if we make this assumption, then the $\widetilde{\mathcal{L}}$ itself, of which we are checking the invariance via the variations above, would be zero. This is due to the fact that $\widetilde{\mathcal{L}}$ is proportional to the equations of motion and checking the invariance of something that is zero is silly. It is true that all our path-integral does is to force us on the classical equations of motion, but still it is better not to use that explicitly.

To avoid that problem the trick to use is to define the following integrated charge:

$$\widetilde{Q_{\scriptscriptstyle H}} = \int_0^T Q_{\scriptscriptstyle H}(t)dt, \tag{A.3}$$

where 0 and T are the endpoints of the interval over which we consider our motion.

It is then easy to check that all steps done in eq.(A.2), once we replace $Q_{\scriptscriptstyle H}$ with $\widetilde{Q_{\scriptscriptstyle H}}$, can go through without assuming the conservation of $Q_{\scriptscriptstyle H}$. In fact the $\frac{d\widetilde{Q_H}}{dt}$ in the third step of equation (A.2) is zero not because of the conservation of $Q_{\scriptscriptstyle H}$ but because $\widetilde{Q_{\scriptscriptstyle H}}$ is independent of t. Moreover the variation δ generated by the $\widetilde{Q_{\scriptscriptstyle H}}$ is the same as the

one generated by Q_H. This is so because in checking the variation induced by $\widetilde{Q_H}$ we have to use the non-equal-time commutators given by the path-integral (6) which are:

$$[\phi^a(t), \lambda_b(t')] = i\delta^a_b \delta(t - t') \tag{A.4}$$

and similarly for the c^a and $\bar{c}_a$.

This charge was introduced before in the literature [12] in order to handle things in an abstract "loop space". In our case we need that charge for the much simpler reasons explained above.

B Appendix

In this appendix we will show what happens when we combine two susy transformations.

Let us define the following two transformations G_{ϵ_1}, G_{ϵ_2} on any of the variables $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$ (which we will collectively indicate with O).

$$\begin{aligned} \delta_1 O \equiv [G_{\epsilon_1}, O] \equiv [\bar{\epsilon}_1 \overline{Q}_H + \epsilon_1 Q_H, O] \\ \delta_2 O \equiv [G_{\epsilon_2}, O] \equiv [\bar{\epsilon}_2 \overline{Q}_H + \epsilon_2 Q_H, O] \end{aligned} \tag{B.1}$$

where the infinitesimal parameters ϵ_1,ϵ_2 are time dependent.

Combining two of these transformations we get

$$[\delta_1, \delta_2] O = [G_{\epsilon_1}, [G_{\epsilon_2}, O]] - [G_{\epsilon_2}, [G_{\epsilon_1}, O]]. \tag{B.2}$$

Applying the Jacobi identity on the RHS of (B.2), we get

$$[\delta_1, \delta_2] O = [[G_{\epsilon_1}, G_{\epsilon_2}], O]. \tag{B.3}$$

By remembering eq. (41) it is easy to work out what $[G_{\epsilon_1}, G_{\epsilon_2}]$ is :

$$[G_{\epsilon_1}, G_{\epsilon_2}] = 2i\beta(\bar{\epsilon}_1 \epsilon_2 + \epsilon_1 \bar{\epsilon}_2)\widetilde{\mathcal{H}}; \tag{B.4}$$

inserting (B.4) in (B.3), we get

$$\begin{aligned} [\delta_1, \delta_2] O &= 2i\beta(\bar{\epsilon}_1 \epsilon_2 + \epsilon_1 \bar{\epsilon}_2)[\widetilde{\mathcal{H}}, O] \\ &= 2\beta(\bar{\epsilon}_1 \epsilon_2 + \epsilon_1 \bar{\epsilon}_2)\frac{dO}{dt}. \end{aligned} \tag{B.5}$$

So we see from here that the composition of two local susy transformations produces a local time-translation with parameter $\bar{\epsilon}_1 \epsilon_2 + \epsilon_1 \bar{\epsilon}_2$.

It is also instructive to do the composition of two *finite* susy transformations, the first (G_1) with parameter ϵ_1 and the other (G_2) with parameter ϵ_2. The transformed variable O' has the expression:

$$O' = \mathrm{e}^{iG_1}\mathrm{e}^{iG_2}\ O\ \mathrm{e}^{-iG_2}\mathrm{e}^{-iG_1}. \tag{B.6}$$

Using the Baker-Hausdorff identity on the RHS above, we obtain:

$$\begin{aligned} O' &= \mathrm{e}^{[iG_1+iG_2-\frac{1}{2}[G_1,G_2]]}\ O\ \mathrm{e}^{[-iG_1-iG_2+\frac{1}{2}[G_1,G_2]]} \\ &= \mathrm{e}^{i[\overline{\epsilon_1}\overline{Q}_H+\epsilon_1 Q_H+\overline{\epsilon_2}\overline{Q}_H+\epsilon_2 Q_H-\beta(\overline{\epsilon_1}\epsilon_2+\epsilon_1\overline{\epsilon_2})\widetilde{\mathcal{H}}]}\ O\ \mathrm{e}^{-i[\overline{\epsilon_1}\overline{Q}_H+\epsilon_1 Q_H+\overline{\epsilon_2}\overline{Q}_H+\epsilon_2 Q_H-\beta(\overline{\epsilon_1}\epsilon_2+\epsilon_1\overline{\epsilon_2})\widetilde{\mathcal{H}}]} \\ &= \mathrm{e}^{i\overline{\gamma}\overline{Q}_H+i\gamma Q_H+i\Delta t\widetilde{\mathcal{H}}}\ O\ \mathrm{e}^{-i\overline{\gamma}\overline{Q}_H-i\gamma Q_H-i\Delta t\widetilde{\mathcal{H}}}, \end{aligned} \tag{B.7}$$

where $\overline{\gamma}$, γ and Δt are respectively

$$\begin{cases} \gamma = \epsilon_1 + \epsilon_2 \\ \overline{\gamma} = \overline{\epsilon_1} + \overline{\epsilon_2} \\ \Delta t = -\beta(\overline{\epsilon_1}\epsilon_2 + \epsilon_1\overline{\epsilon_2}). \end{cases} \tag{B.8}$$

So we see from eq.(B.7) that the composition of two finite local susy is a local susy plus a local time-translation.

We will write down here how the variables $(\varphi^a, c^a, \lambda_a, \overline{c}_a)$ transform under a local time translation:

$$\begin{cases} \delta^{loc}_{\widetilde{\mathcal{H}}}\phi^a = [\eta(t)\widetilde{\mathcal{H}}, \phi^a] = -i\eta\ \omega^{an}\partial_n\widetilde{\mathcal{H}} \\ \delta^{loc}_{\widetilde{\mathcal{H}}}\lambda_a = [\eta(t)\widetilde{\mathcal{H}}, \lambda_a] = i\eta\ \partial_a\widetilde{\mathcal{H}} \\ \delta^{loc}_{\widetilde{\mathcal{H}}}c^a = [\eta(t)\widetilde{\mathcal{H}}, c^a] = -i\eta\ \omega^{an}\partial_n\partial_l Hc^l \\ \delta^{loc}_{\widetilde{\mathcal{H}}}\bar{c}_a = [\eta(t)\widetilde{\mathcal{H}}, \bar{c}_a] = i\eta\ \bar{c}_m\omega^{mn}\partial_n\partial_a H. \end{cases} \tag{B.9}$$

C Appendix

In this appendix we analyze the question of how the transformations (55) are generated by our first class constraints (56)(61). This is a delicate issue which is explained in detail on page 75 ff of ref. [11]. In fact naively the transformation on g contained in (55) apparently cannot be obtained by doing the commutator of g with the proper gauge generators. The authors of ref. [11] are aware of similar problems and they suggested the following approach. First let us build an extended action defined in the following way:

$$S_{ext.} = \int dt[\widetilde{\mathcal{L}}_{susy} + \Pi_\psi\dot{\psi} + \Pi_{\overline{\psi}}\dot{\overline{\psi}} + \Pi_g\dot{g} - U^{(i)}G_i] \tag{C.1}$$

where the G_i are all the six first class constraints (65) (and not just the primary ones) and the $U^{(i)}$ the relative Lagrange multipliers. A general gauge transformation on an observable O will be:

$$\delta O = [\overline{\epsilon}\overline{Q}_{H} + \epsilon Q_{H} + \eta\widetilde{\mathcal{H}} + \overline{\alpha}\Pi_{\psi} + \alpha\Pi_{\overline{\psi}} + \beta\Pi_{g}, O] \tag{C.2}$$

where $(\overline{\epsilon}, \epsilon, \eta, \overline{\alpha}, \alpha, \beta)$ are six infinitesimal gauge parameters associated to the six generators G_i. If we consider the Lagrange multipliers U^i as functions of the basic variables, then they will also change under the gauge transformation above. As we do not know the exact expression of the U^i in terms of the basic variables, we will formally indicate their gauge variation as δU^i. Using this notation it is then a simple but long calculation to show that the gauge variation of the action $S_{ext.}$ is:

$$\begin{aligned}\delta S_{ext} = \int dt \quad \Big[& i\dot{\epsilon}Q_{H} - i\dot{\overline{\epsilon}}\overline{Q}_{H} - i\dot{\eta}\widetilde{\mathcal{H}} - 2i\widetilde{\mathcal{H}}(\epsilon\psi + \overline{\epsilon}\overline{\psi}) + \\ & + \overline{\alpha}\overline{Q}_{H} + \alpha Q_{H} - i\beta\widetilde{\mathcal{H}} + \Pi_{\psi}\dot{\overline{\alpha}} + \Pi_{\overline{\psi}}\dot{\alpha} - i\Pi_{g}\dot{\beta} + \\ & - \delta U^{(2)}\Pi_{\psi} - \delta U^{(1)}\Pi_{\overline{\psi}} - \delta U^{(3)}\Pi_{g} - \delta U^{(4)}Q_{H} + \\ & - \delta U^{(5)}\overline{Q}_{H} - \delta U^{(6)}\widetilde{\mathcal{H}} - U^{(4)}\overline{\epsilon}2i\widetilde{\mathcal{H}} - U^{(5)}\epsilon 2i\widetilde{\mathcal{H}}\Big],\end{aligned} \tag{C.3}$$

where we have indicated with $U^{(1)}$, for example, the Lagrange multiplier associated to the first of the constraints in (65), with $U^{(2)}$ the one associated to the second and so on.

It is now easy to choose the variation of the Lagrange multipliers in such a way to make $\delta S_{ext} = 0$:

$$\left\{\begin{array}{l}\delta U^{(1)} = -\dot{\alpha} \\ \delta U^{(2)} = -\dot{\overline{\alpha}} \\ \delta U^{(3)} = -i\dot{\beta}\end{array}\right. \quad ; \quad \left\{\begin{array}{l}\delta U^{(4)} = \alpha - i\dot{\epsilon} \\ \delta U^{(5)} = \overline{\alpha} - i\dot{\overline{\epsilon}} \\ \delta U^{(6)} = -i\dot{\eta} - i\beta - 2i(\overline{\epsilon}\overline{\psi} + \epsilon\psi) + 2i(\overline{\epsilon}U^{(4)} + \epsilon U^{(5)}).\end{array}\right. \tag{C.4}$$

We can now proceed as in ref. [11] by restricting the Lagrange multipliers to be only those of the primary fields (56)

$$U^{(4)} = U^{(5)} = U^{(6)} = 0 \tag{C.5}$$

which implies that the gauge variations of these must be zero. From these two conditions we get from eq.(C.4) the following relations among the six gauge parameters

$$\left\{\begin{array}{l}\alpha = i\dot{\epsilon} \\ \overline{\alpha} = i\dot{\overline{\epsilon}} \\ \beta = -\dot{\eta} - 2(\overline{\epsilon}\overline{\psi} + \epsilon\psi).\end{array}\right. \tag{C.6}$$

As a consequence the general gauge variation of an observable O given in eq.(C.2) becomes

$$\delta O = [\overline{\epsilon}\overline{Q}_{H} + \epsilon Q_{H} + \eta\widetilde{\mathcal{H}} + i\dot{\overline{\epsilon}}\Pi_{\psi} + i\dot{\epsilon}\Pi_{\overline{\psi}} + (-\dot{\eta} - 2(\overline{\epsilon}\overline{\psi} + \epsilon\psi))\Pi_{g}, O] \tag{C.7}$$

and applying it on the three variables $(\psi, \overline{\psi}, g)$ we get:

$$\begin{cases} \delta\psi = [i\dot{\overline{\epsilon}}\Pi_{\psi}, \psi] = i\dot{\overline{\epsilon}} \\ \delta\overline{\psi} = [i\dot{\epsilon}\Pi_{\overline{\psi}}, \overline{\psi}] = i\dot{\epsilon} \\ \delta g = [-(\dot{\eta} + 2(\overline{\epsilon}\overline{\psi} + \epsilon\psi)\Pi_{g}, g] = i\dot{\eta} + 2i(\overline{\epsilon}\overline{\psi} + \epsilon\psi). \end{cases} \tag{C.8}$$

This is exactly the transformation (55) obtained here from the generators G_i of eq.(65). This concludes the explanation of how the variations (55), derived from a pure lagrangian variation, could be obtained via the *canonical* gauge generators G_i.

D Appendix

In this appendix, for purely pedagogical reasons, we will show how to gauge away the $(\psi, \overline{\psi}, g)$.

The infinitesimal transformations are given in eq.(55) and the first thing to do is to build finite transformations out of the infinitesimal ones. If we start from a configuration $(\psi_0(t), \overline{\psi}_0(t), g_0(t))$, after one step we arrive at $(\psi_1(t), \overline{\psi}_1(t), g_1(t))$ which are given by:

$$\begin{cases} \psi_1(t) = \psi_0(t) + i\dot{\overline{\epsilon}}(t) \\ \overline{\psi}_1(t) = \overline{\psi}_0(t) + i\dot{\epsilon}(t) \\ g_1(t) = g_0(t) + i\dot{\eta}(t) + 2i(\epsilon(t)\psi_0(t) + \overline{\epsilon}(t)\overline{\psi}_0(t)). \end{cases} \tag{D.1}$$

It is not difficult to work out what we get after N steps:

$$\begin{cases} \psi_N(t) = \psi_0(t) + iN\dot{\overline{\epsilon}}(t) \\ \overline{\psi}_N(t) = \overline{\psi}_0(t) + iN\dot{\epsilon}(t) \\ g_N(t) = g_0(t) + iN\dot{\eta}(t) + 2i(N\epsilon(t)\psi_0(t) + N\overline{\epsilon}(t)\overline{\psi}_0(t)) - N(N+1)(\epsilon\dot{\overline{\epsilon}} + \overline{\epsilon}\dot{\epsilon}). \end{cases} \tag{D.2}$$

Taking now the limit $N \to \infty$, but with the conditions:

$$\begin{cases} N\epsilon(t) \longrightarrow \Delta(t) \\ N\overline{\epsilon}(t) \longrightarrow \overline{\Delta}(t) \\ N\eta(t) \longrightarrow \Delta_g(t), \end{cases} \tag{D.3}$$

where the various $\Delta(t)$ are non-divergent quantities, we get that a finite transformation has the form:

$$\begin{cases} \psi(t) = \psi_0(t) + i\dot{\overline{\Delta}}(t) \\ \overline{\psi}(t) = \overline{\psi}_0(t) + i\dot{\Delta}(t) \\ g(t) = g_0(t) + i\dot{\Delta}_g(t) + 2i(\Delta(t)\psi_0(t) + \overline{\Delta}(t)\overline{\psi}_0(t)) - (\Delta\dot{\overline{\Delta}} + \overline{\Delta}\dot{\Delta}). \end{cases} \tag{D.4}$$

From the equation above it is easy to see that with the following choice of Δ's

$$\left\{\begin{array}{l} \overline{\Delta}(t)=i\displaystyle\int_0^t\psi_0(\tau)d\tau \\ \Delta(t)=i\displaystyle\int_0^t\overline{\psi}_0(\tau)d\tau \\ \Delta_g(t)=\displaystyle\int_0^t d\tau[ig_0(\tau)-2(\Delta\psi_0+\overline{\Delta\psi}_0)-i(\Delta\dot{\overline{\Delta}}+\overline{\Delta}\dot{\Delta})] \end{array}\right. \tag{D.5}$$

we can bring the $(\psi,\overline{\psi},g)$ to zero. We should anyhow be careful and check whether there are no obstruction to this construction. Actually, after eq.(46) we said that, in order that the surface terms disappear, we needed to require that $\epsilon(t)$ and $\overline{\epsilon}(t)$ be zero at the end-points $(0,T)$ of integration. From eq.(D.3) one sees that this implies:

$$\begin{array}{l} \Delta(0)=\overline{\Delta}(0)=0 \\ \Delta(T)=\overline{\Delta}(T)=0. \end{array} \tag{D.6}$$

While the first condition is easily satisfied, as can be seen from eq.(D.5), the second one would imply:

$$\begin{array}{l} \Delta(T)=i\displaystyle\int_0^T\overline{\psi}_0(\tau)d\tau=0 \\ \overline{\Delta}(T)=i\displaystyle\int_0^T\psi_0(\tau)d\tau=0. \end{array} \tag{D.7}$$

This is a condition which is not satisfied by any initial configuration ψ_0, $\overline{\psi}_0$ but only by special ones. So we can say that, if we want transformations which do not leave surface terms, then it may be impossible to gauge away $(\psi,\overline{\psi})$. Not to have surface terms may turn out to be an important issue in some contexts. Anyhow this problem does not arise in the time-reparametrization transformation because, as we see from eq.(52), that transformation does not generate surface terms.

E Appendix

In this appendix we will show how the constraints (87) affect the Hilbert space of the system. We know that the *physical* states should be annihilated by the constraints:

$$\left\{\begin{array}{l} [\Pi_\alpha-Q_{BRS}]\mid\text{phys}\rangle=0 \\ [\Pi_{\overline{\alpha}}-\overline{Q}_{BRS}]\mid\text{phys}\rangle=0 \end{array}\right. \tag{E.1}$$

and so this seems to restrict the original Hilbert space of the system. On the other hand we have proved that the system obeying these constraints and with lagrangian (85) has the same number of degrees of freedom as the original system with lagrangian (7) and moreover they seem equivalent. If that is so then the Hilbert space of the

physical states should be equivalent or isomorphic to the original Hilbert space. This is what we are going to prove in what follows.

Let us first solve the constraint (E.1). The wave-functions $\Psi(\cdots)$ of the system will depend not only on the (φ^a, c^a) but also on the gauge-parameters $\alpha(t)$ and $\overline{\alpha}(t)$. So equation (E.1) takes the form:

$$\begin{cases} \dfrac{\partial\Psi(\varphi,c;\alpha,\overline{\alpha})}{\partial\alpha} = Q_{BRS}\Psi(\varphi,c;\alpha,\overline{\alpha}) \\ \dfrac{\partial\Psi(\varphi,c;\alpha,\overline{\alpha})}{\partial\overline{\alpha}} = \overline{Q}_{BRS}\Psi(\varphi,c;\alpha,\overline{\alpha}) \end{cases} \tag{E.2}$$

whose solution is

$$\Psi(\varphi,c;\alpha,\overline{\alpha}) = \exp[\alpha Q_{BRS} + \overline{\alpha}\overline{Q}_{BRS}]\ \psi(\varphi,c) \tag{E.3}$$

where the $\psi(\varphi,c)$ are the states of the Hilbert space of the old system with lagrangian (7) and the Q_{BRS} and $\overline{Q}_{BRS}$ should be interpreted as the differential operator associated to the relative charge via the substitution (17); the same for all the Hamiltonians which we will use from now on. To prove that the two systems are equivalent we should prove that there is an isomorphism in Hilbert space between the solutions of the two Koopman-von Neumann[4] equations, the first one associated to the old Hamiltonian (12) and the second to the Hamiltonian of the lagrangian (85). This last one is the *primary* [9] Hamiltonian:

$$\widetilde{\mathcal{H}}_P \equiv \widetilde{\mathcal{H}}_{can} + \mu(\Pi_\alpha - Q_{BRS}) + \overline{\mu}(\Pi_{\overline{\alpha}} - \overline{Q}_{BRS}) \tag{E.4}$$

where μ and $\overline{\mu}$ are Lagrange multipliers and $\widetilde{\mathcal{H}}_{can}$ is the *canonical* [9] Hamiltonian associated to the Lagrangian (85).

The Koopman-von Neumann equation for this system is:

$$\widetilde{\mathcal{H}}_P\Psi(\varphi,c,t;\alpha,\overline{\alpha}) = i\frac{\partial\Psi(\varphi,c,t;\alpha,\overline{\alpha})}{\partial t} \tag{E.5}$$

which can be rewritten as:

$$[\widetilde{\mathcal{H}} + \mu(\Pi_\alpha - Q_{BRS}) + \overline{\mu}(\Pi_{\overline{\alpha}} - \overline{Q}_{BRS})]\Psi(\varphi,c,t;\alpha,\overline{\alpha}) = i\frac{\partial\Psi(\varphi,c,t;\alpha,\overline{\alpha})}{\partial t}. \tag{E.6}$$

Since $\Psi(\varphi,c,t;\alpha,\overline{\alpha})$ is annihilated by the constraints (E.1) we get:

$$\widetilde{\mathcal{H}}\Psi(\varphi,c,t;\alpha,\overline{\alpha}) = i\frac{\partial\Psi(\varphi,c,t;\alpha,\overline{\alpha})}{\partial t}; \tag{E.7}$$

[4] By Koopman von Neumann equation we mean the analog of the Liouville equations built via the full $\widetilde{\mathcal{H}}$ and not via just its bosonic part.

now we use (E.3) in (E.7) and this yields:

$$\widetilde{\mathcal{H}}\exp[\alpha Q_{BRS}+\overline{\alpha}\overline{Q}_{BRS}]\psi(\varphi,c,t)=i\frac{\partial}{\partial t}\Big[\exp(\alpha Q_{BRS}+\overline{\alpha}\overline{Q}_{BRS})\psi(\varphi,c,t)\Big]. \tag{E.8}$$

The last step is to work out the derivatives in eq.(E.8); we obtain:

$$\exp[\alpha Q_{BRS}+\overline{\alpha}\overline{Q}_{BRS}]\ \widetilde{\mathcal{H}}\psi(\varphi,c,t)=\exp[\alpha Q_{BRS}+\overline{\alpha}\overline{Q}_{BRS}]\ i\frac{\partial\psi(\varphi,c,t)}{\partial t} \tag{E.9}$$

which holds iff

$$\widetilde{\mathcal{H}}\psi=i\frac{\partial\psi}{\partial t} \tag{E.10}$$

and this concludes the proof that the two systems have not only the same number of degrees of freedom but also the same Hilbert space.

F Appendix

In this appendix we provide details regarding the derivation of eq. (108). Consider first the dependence on g. From

$$|\,\mathrm{phys}\rangle=Q_{(1)}|\chi\rangle\quad\text{and}\quad\Pi_g\,|\,\mathrm{phys}\rangle=0 \tag{F.1}$$

we infer that

$$\Pi_g Q_{(1)}|\chi\rangle=Q_{(1)}\Pi_g|\chi\rangle=0, \tag{F.2}$$

which means that

$$\Pi_g|\chi\rangle\in\ker Q_{(1)}. \tag{F.3}$$

If we represent Π_g as $\Pi_g=-i\frac{\partial}{\partial g}$, eq.(F.3) implies:

$$\frac{\partial}{\partial g}|\chi\rangle=\sum_m f_m(g,\alpha)|\zeta_m\rangle \tag{F.4}$$

where $|\zeta_m\rangle$ form a basis of $\ker Q_{(1)}$. Solving this last differential equation we get

$$|\chi\rangle=|\chi_0;\alpha\rangle+\sum_m\left[\int dg\ f_m(g,\alpha)\right]|\zeta_m\rangle \tag{F.5}$$

where $= |\chi_0;\alpha\rangle$ does not depend on g anymore. By the same line of reasoning we can prove that $\Pi_\alpha|\chi\rangle \in \ker Q_{(1)}$, which in turn implies that $\Pi_\alpha|\chi_0;\alpha\rangle \in \ker Q_{(1)}$. We can repeat the previous steps and we arrive at the relation:

$$|\chi_0;\alpha\rangle = |\chi_0\rangle + \sum_m \left[\int d\alpha\ l_m(g,\alpha)\right]|\zeta_m\rangle \tag{F.6}$$

which, substituted in eq. (F.5), yields:

$$|\chi\rangle = |\chi_0\rangle + \sum_m \left[\int d\alpha\ l_m(g,\alpha)\right]|\zeta_m\rangle + \sum_m \left[\int dg\ f_m(g,\alpha)\right]|\zeta_m\rangle \equiv |\chi_0\rangle + |\zeta;\alpha,g\rangle, \tag{F.7}$$

as we claimed in eq. (108).

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